

**Trinity Western University**  
**Department of Mathematical Sciences**  
**MATH250 (Linear Algebra)**  
**Mid-Term I Examination Solution**

1. Discuss the solution of the system of equations

$$\begin{aligned}
 x - 4y + 2z &= 2 \\
 3x + (a - 4)y &= 1 \\
 3x - y + (a - 3)z &= 1
 \end{aligned}$$

for various values of  $a$  (Indicate in each case how many solutions will you get, also giving the solutions if they exist).

**Solution:**

The augmented matrix is

$$\begin{aligned}
 &\left( \begin{array}{cccc} 1 & -4 & 2 & 2 \\ 3 & a-4 & 0 & 1 \\ 3 & -1 & a-3 & 1 \end{array} \right) && R_{12}(-3), R_{13}(-3) \\
 \sim &\left( \begin{array}{cccc} 1 & -4 & 2 & 2 \\ 0 & a+8 & -6 & -5 \\ 0 & 11 & a-9 & -5 \end{array} \right) && R_{23} \\
 \sim &\left( \begin{array}{cccc} 1 & -4 & 2 & 2 \\ 0 & 11 & a-9 & -5 \\ 0 & a+8 & -6 & -5 \end{array} \right) && R_{23}\left(-\frac{a+8}{11}\right) \\
 \sim &\left( \begin{array}{cccc} 1 & -4 & 2 & 2 \\ 0 & 11 & a-9 & -5 \\ 0 & 0 & -6 - \frac{1}{11}(a-9)(a+8) & -5 + \frac{5}{11}(a+8) \end{array} \right) \\
 = &\left( \begin{array}{cccc} 1 & -4 & 2 & 2 \\ 0 & 11 & a-9 & -5 \\ 0 & 0 & -\frac{1}{11}(a^2 - a - 6) & \frac{5}{11}(a-3) \end{array} \right) && R_3(-11) \\
 \sim &\left( \begin{array}{cccc} 1 & -4 & 2 & 2 \\ 0 & 11 & a-9 & -5 \\ 0 & 0 & (a-3)(a+2) & -5(a-3) \end{array} \right)
 \end{aligned}$$

We look for the first non-zero entry in the third row. It is  $(a-3)(a+2)$ , unless it is zero. Thus we must make a distinction between the two cases (i)  $(a-3)(a+2) \neq 0$ , and  $(a-3)(a+2) = 0$ .

Case (i)  $(a-3)(a+2) \neq 0$ , i.e.,  $a \neq 3$  and  $a \neq -2$ . Performing  $R_3\left(\frac{1}{(a-3)(a+2)}\right)$

on the augmented matrix, we get

$$\left( \begin{array}{cccc} 1 & -4 & 2 & 2 \\ 0 & 11 & a-9 & -5 \\ 0 & 0 & 1 & -\frac{5}{a+2} \end{array} \right)$$

The given system is equivalent to

$$\begin{aligned}x - 4y + 2z &= 2 \\11y + (a - 9)z &= -5 \\z &= -\frac{5}{a + 2}\end{aligned}$$

Performing back-substitution we get

$$y = \frac{1}{11}[-5 - (a - 9)z] = \frac{1}{11} \left[ -5 + \frac{5(a - 9)}{a + 2} \right] = -\frac{5}{a + 2}$$

$$x = 2 + 4y - 2z = 2 - \frac{10}{a + 2} = \frac{2(a - 3)}{a + 2}$$

So in this case we get the unique solution

$$x = \frac{2(a - 3)}{a + 2}, \quad y = -\frac{5}{a + 2}, \quad z = -\frac{5}{a + 2}$$

Case (ii)  $(a - 3)(a + 2) = 0$ . Now the augmented matrix becomes

$$\begin{pmatrix} 1 & -4 & 2 & 2 \\ 0 & 11 & a - 9 & -5 \\ 0 & 0 & 0 & -5(a - 3) \end{pmatrix}$$

Looking for the first non-zero entry, we find that it is  $-5(a - 3)$ , unless it is equal to zero. Therefore, again, we must make a distinction, between the two cases (iia)  $a - 3 \neq 0$ , and (iib)  $a - 3 = 0$ .

Case (iia)  $a - 3 \neq 0$ , and  $(a - 3)(a + 2) = 0$ , i.e.,  $a \neq 3$ , and  $a = 3$  or  $a = -2 \Rightarrow a = -2$  (since  $a = 3$  and  $a \neq 3$  at the same time is impossible). For this value of  $a$ , the augmented matrix takes the form

$$\begin{pmatrix} 1 & -4 & 2 & 2 \\ 0 & 11 & -11 & -5 \\ 0 & 0 & 0 & 25 \end{pmatrix} \quad R_2 \left( \frac{1}{11} \right), \quad R_3 \left( \frac{1}{25} \right) \\ \sim \begin{pmatrix} 1 & -4 & 2 & 2 \\ 0 & 1 & -1 & -\frac{5}{11} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is the row echelon form of the starting augmented matrix. The last row leads to

$$0x + 0y + 0z = 1$$

No values of  $x$ ,  $y$  and  $z$  can satisfy the above equation. Hence when  $a = -2$ , there is no solution.

Case (iib)  $a - 3 = 0$ , and  $(a - 3)(a + 2) = 0 \Rightarrow a = 3$ . Now the augmented matrix becomes

$$\begin{pmatrix} 1 & -4 & 2 & 2 \\ 0 & 11 & -6 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The equivalent system of equations is

$$x - 4y + 2z = 2, \quad 11y - 6z = -5, \quad 0 = 0$$

Solving for the leading variables we obtain

$$x = 2 + 4y - 2z, \quad y = \frac{1}{11}(-5 + 6z)$$

Setting the free variable  $z = t$ , the following infinitely many solutions are obtained

$$x = \frac{2}{11} + \frac{2}{11}t, y = -\frac{5}{11} + \frac{6}{11}t, z = t, t \in \mathbb{R}$$

We can summarize the results:

When  $a = -2$ , there is no solution.

When  $a = 3$ , there are infinitely many solutions.

Otherwise there is an unique solution.

2. Show that if  $A$  and  $B$  are  $3 \times 3$  matrices then  $\text{tr}(AB) = \text{tr}(BA)$ . Use this fact to show that in general, i.e., for any two  $n \times n$  matrices  $A$  and  $B$ ,  $AB - BA = I_n$  is not possible.

**Solution:**

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

Then

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}$$

and

$$BA = \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} + b_{13}a_{31} & b_{11}a_{12} + b_{12}a_{22} + b_{13}a_{32} & b_{11}a_{13} + b_{12}a_{23} + b_{13}a_{33} \\ b_{21}a_{11} + b_{22}a_{21} + b_{23}a_{31} & b_{21}a_{12} + b_{22}a_{22} + b_{23}a_{32} & b_{21}a_{13} + b_{22}a_{23} + b_{23}a_{33} \\ b_{31}a_{11} + b_{32}a_{21} + b_{33}a_{31} & b_{31}a_{12} + b_{32}a_{22} + b_{33}a_{32} & b_{31}a_{13} + b_{32}a_{23} + b_{33}a_{33} \end{pmatrix}$$

Therefore

$$\text{tr}(AB) = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33}$$

$$= \sum_{i=1}^3 \sum_{j=1}^3 a_{ij}b_{ji}$$

and

$$\text{tr}(BA) = b_{11}a_{11} + b_{12}a_{21} + b_{13}a_{31} + b_{21}a_{12} + b_{22}a_{22} + b_{23}a_{32} + b_{31}a_{13} + b_{32}a_{23} + b_{33}a_{33}$$

$$= \sum_{i=1}^3 \sum_{j=1}^3 b_{ij}a_{ji} = \sum_{i=1}^3 \sum_{j=1}^3 a_{ij}b_{ji}$$

Clearly

$$\text{tr}(AB) = \text{tr}(BA)$$

and the result is also valid for  $n \times n$  matrices.

For  $AB - BA = I_n$  to be true,  $\text{tr}(AB - BA) = \text{tr}(I_n)$

But  $\text{tr}(AB - BA) = \text{tr}(AB) - \text{tr}(BA) = 0$ , and  $\text{tr}(I_n) = n$ , and the two cannot be equal. Thus  $AB - BA = I_n$  is clearly impossible.

3. Find an  $LU$ -decomposition for  $A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{pmatrix}$

**Solution:**

We have

$$\begin{aligned}
 A &= \begin{pmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{pmatrix} && R_{12}(-\frac{1}{2}), R_{13}(\frac{1}{2}) \\
 &\sim \begin{pmatrix} 2 & 4 & 2 \\ \boxed{\frac{1}{2}} & -1 & 1 \\ \boxed{-\frac{1}{2}} & 2 & 3 \end{pmatrix} && R_{23}(2) \\
 &\sim \begin{pmatrix} 2 & 4 & 2 \\ \boxed{\frac{1}{2}} & -1 & 1 \\ \boxed{-\frac{1}{2}} & \boxed{-2} & 5 \end{pmatrix},
 \end{aligned}$$

the numbers in the boxes being the multipliers.

Hence

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{pmatrix} \text{ and } U = \begin{pmatrix} 2 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

Of course, the  $LU$ -decomposition can be done in other ways also, e.g.,

$$\begin{aligned}
 A = LU &= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 5 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}
 \end{aligned}$$

$$\text{and now } L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{pmatrix}, U = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Compute  $\det(A)$  if

$$\det \begin{pmatrix} a & b & c \\ p & q & r \\ x & y & z \end{pmatrix} = 7, \text{ and } A = \begin{pmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{pmatrix}$$

**Solution:**

We have

$$\det(A) = \begin{vmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{vmatrix}$$

$$\begin{aligned}
&= \begin{vmatrix} 2a & 2b & 2c \\ 2p & 2q & 2r \\ 2x & 2y & 2z \end{vmatrix} + \begin{vmatrix} 2a & 2b & 2c \\ 2p & 2q & 2r \\ a & b & c \end{vmatrix} + \begin{vmatrix} 2a & 2b & 2c \\ x & y & z \\ 2x & 2y & 2z \end{vmatrix} + \begin{vmatrix} 2a & 2b & 2c \\ x & y & z \\ a & b & c \end{vmatrix} + \\
&+ \begin{vmatrix} p & q & r \\ 2p & 2q & 2r \\ 2x & 2y & 2z \end{vmatrix} + \begin{vmatrix} p & q & r \\ 2p & 2q & 2r \\ a & b & c \end{vmatrix} + \begin{vmatrix} p & q & r \\ x & y & z \\ 2x & 2y & 2z \end{vmatrix} + \begin{vmatrix} p & q & r \\ x & y & z \\ a & b & c \end{vmatrix}
\end{aligned}$$

Except for the first and the last, all other determinants become zero, because they have proportional rows.

Hence

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 2a & 2b & 2c \\ 2p & 2q & 2r \\ 2x & 2y & 2z \end{vmatrix} + \begin{vmatrix} p & q & r \\ x & y & z \\ a & b & c \end{vmatrix} \\
&= 8 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} - \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}, \text{ on interchanging the first and third rows} \\
&= 8 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}, \text{ on interchanging the second and third rows} \\
&= 9 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 9 \times 7 = 63.
\end{aligned}$$

Alternately we have

$$\begin{aligned}
A &= \begin{pmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} a & b & c \\ p & q & r \\ x & y & z \end{pmatrix} \\
\Rightarrow \det(A) &= \det \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} \det \begin{pmatrix} a & b & c \\ p & q & r \\ x & y & z \end{pmatrix}
\end{aligned}$$

$$\text{But } \det \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} = 2 \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = 2(4) - (-1) = 9$$

$$\text{and } \det \begin{pmatrix} a & b & c \\ p & q & r \\ x & y & z \end{pmatrix} = 7 \text{ (given)}$$

$$\text{Hence } \det(A) = 9 \times 7 = 63$$

We can also solve it as under.

We have

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{vmatrix} && R_{12}(-2) \\
&= \begin{vmatrix} 2a+p & 2b+q & 2c+r \\ x-4a & y-4b & z-4c \\ 2x+a & 2y+b & 2z+c \end{vmatrix} && R_{23}(-2)
\end{aligned}$$

$$\begin{aligned}
&= \begin{vmatrix} 2a+p & 2b+q & 2c+r \\ x-4a & y-4b & z-4c \\ 9a & 9b & 9c \end{vmatrix} \\
&= 9 \begin{vmatrix} 2a+p & 2b+q & 2c+r \\ x-4a & y-4b & z-4c \\ a & b & c \end{vmatrix} \quad R_{31}(-2), R_{32}(4) \\
&= 9 \begin{vmatrix} p & q & r \\ x & y & z \\ a & b & c \end{vmatrix} = 9 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 9 \times 7 = 63
\end{aligned}$$

5. A lab rat has a choice of three foods  $P$ ,  $Q$  and  $R$  each day. On any given day it has a 60% chance of choosing the same food as it chose the previous day, and is equally likely to choose either of the other foods.

- If it chooses  $P$  one day, find the probability it chooses  $Q$  three days later.
- What percentage of its meals are food  $P$ ,  $Q$  and  $R$ ?

**Solution:**

The transition matrix is

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}$$

(a) We have  $\mathbf{x}^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ , then

$$\mathbf{x}^{(1)} = P\mathbf{x}^{(0)} = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.2 \\ 0.2 \end{pmatrix}$$

$$\mathbf{x}^{(2)} = P\mathbf{x}^{(1)} = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.2 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.44 \\ 0.28 \\ 0.28 \end{pmatrix}$$

$$\mathbf{x}^{(3)} = P\mathbf{x}^{(2)} = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix} \begin{pmatrix} 0.44 \\ 0.28 \\ 0.28 \end{pmatrix} = \begin{pmatrix} 0.376 \\ 0.312 \\ 0.312 \end{pmatrix}$$

Thus the probability of the rat choosing  $Q$  (or  $R$ ) three days later is 0.312.

(b) Let the steady state be  $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$

We have

$$\begin{aligned}
P\mathbf{q} = \mathbf{q} &\Rightarrow \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \\
&\Rightarrow 0.6q_1 + 0.2q_2 + 0.2q_3 = q_1, \quad 0.2q_1 + 0.6q_2 + 0.2q_3 = q_2, \quad 0.2q_1 + 0.2q_2 + 0.6q_3 = q_3. \\
&\Rightarrow -0.4q_1 + 0.2q_2 + 0.2q_3 = 0, \quad 0.2q_1 - 0.4q_2 + 0.2q_3 = 0, \quad 0.2q_1 + 0.2q_2 - 0.4q_3 = 0.
\end{aligned}$$

$$\Rightarrow -2q_1 + q_2 + q_3 = 0, \quad q_1 - 2q_2 + q_3 = 0, \quad q_1 + q_2 - 2q_3 = 0$$

Solving we get

$$\frac{q_1}{1} = \frac{q_2}{1} = \frac{q_3}{1} = \frac{q_1 + q_2 + q_3}{3} = \frac{1}{3}$$

$$\Rightarrow q_1 = q_2 = q_3 = \frac{1}{3},$$

a result to be expected, as the rat does not show any preference for any of the three types of food.