

Trinity Western University
Department of Mathematical Sciences
MATH250 (Linear Algebra)
Mid-Term II Examination Solution

1. Show that if \mathbf{u} and \mathbf{v} are **unit** vectors then the vector $\mathbf{u} + \mathbf{v}$ bisects the direction of the vectors \mathbf{u} and \mathbf{v} .

Use the above fact to find the equations of the lines bisecting the two intersecting lines

$$L_1: (x, y, z) = (-1, 0, 2) + t(2, 2, 1) \text{ and } L_2: (x, y, z) = (-1, 0, 2) + t(6, -2, 3)$$

Solution:

Let $\mathbf{u} + \mathbf{v}$ make angles α and β with \mathbf{u} and \mathbf{v} respectively.

We have

$$\cos \alpha = \frac{(\mathbf{u} + \mathbf{v}) \cdot \mathbf{u}}{|\mathbf{u} + \mathbf{v}| |\mathbf{u}|} = \frac{\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u}}{|\mathbf{u} + \mathbf{v}| |\mathbf{u}|} = \frac{1 + \mathbf{u} \cdot \mathbf{v}}{|\mathbf{u} + \mathbf{v}|}, \text{ since } \mathbf{u} \text{ is a unit vector,}$$

$$\cos \beta = \frac{(\mathbf{u} + \mathbf{v}) \cdot \mathbf{v}}{|\mathbf{u} + \mathbf{v}| |\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}}{|\mathbf{u} + \mathbf{v}| |\mathbf{v}|} = \frac{1 + \mathbf{u} \cdot \mathbf{v}}{|\mathbf{u} + \mathbf{v}|}, \text{ since } \mathbf{v} \text{ is a unit vector.}$$

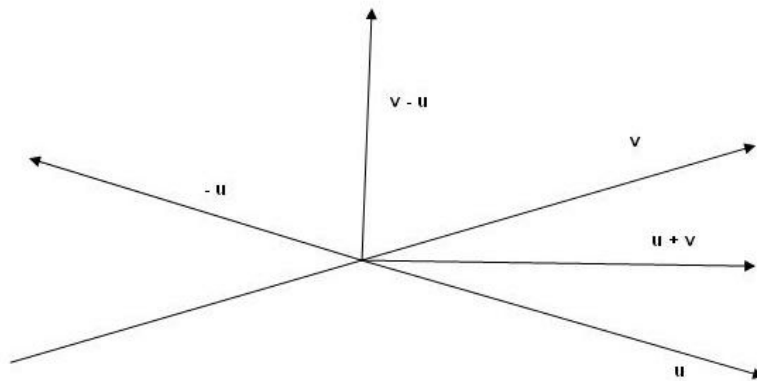
Since $\cos \alpha = \cos \beta$, it follows that $\alpha = \beta$, and $\mathbf{u} + \mathbf{v}$ bisects the direction of the vectors \mathbf{u} and \mathbf{v} .

The direction vectors along the two lines are $\mathbf{u} = (2, 2, 1)$, and $\mathbf{v} = (6, -2, 3)$. If we wish to use the above fact, we require these vectors to be **unit** vectors (i.e., of unit length). They can be readily converted to unit vectors by dividing them by their respective lengths. Thus we have

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{\sqrt{2^2 + 2^2 + 1^2}}(2, 2, 1) = \frac{1}{3}(2, 2, 1) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{6^2 + (-2)^2 + 3^2}}(6, -2, 3) = \frac{1}{7}(6, -2, 3) = \left(\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}\right)$$

The two bisectors have the direction vectors $\hat{\mathbf{u}} + \hat{\mathbf{v}}$ and $\hat{\mathbf{v}} - \hat{\mathbf{u}}$ (see the diagram below).



$$\hat{\mathbf{u}} + \hat{\mathbf{v}} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) + \left(\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}\right) = \left(\frac{32}{21}, \frac{8}{21}, \frac{16}{21}\right)$$

$$\hat{\mathbf{v}} - \hat{\mathbf{u}} = \left(\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}\right) - \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) = \left(\frac{4}{21}, -\frac{20}{21}, \frac{2}{21}\right)$$

There is no loss of generality in taking these vectors as $(4, 1, 2)$ and $(2, -10, 1)$ respectively.

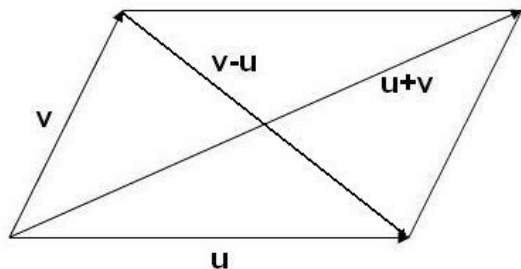
Hence the two bisectors are

$$(x, y, z) = (-1, 0, 2) + t(4, 1, 2) \text{ and } (x, y, z) = (-1, 0, 2) + t(2, -10, 1)$$

2. Show that if the diagonals of a parallelogram are perpendicular, it is necessarily a rhombus.

Solution:

Let the two adjacent sides of the parallelogram be \mathbf{u} and \mathbf{v} . Then the two diagonals are $\mathbf{u} + \mathbf{v}$ and $\mathbf{v} - \mathbf{u}$. (See diagram below)



If they are perpendicular to each other, we must have

$$\begin{aligned} (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{v} - \mathbf{u}) &= 0 \\ \Rightarrow \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{u} &= 0 \\ \Rightarrow |\mathbf{v}|^2 - |\mathbf{u}|^2 &= 0 \\ \Rightarrow |\mathbf{v}|^2 &= |\mathbf{u}|^2 \\ \Rightarrow |\mathbf{v}| &= |\mathbf{u}| \end{aligned}$$

Since the lengths of the adjacent sides of the parallelogram are equal, it must be a rhombus.

3. For what value(s) of k the range of the linear operator defined by the equations

$$\begin{aligned} w_1 &= x_1 + x_2 + 5x_3 \\ w_2 &= x_1 + 2x_2 + 7x_3 \\ w_3 &= 2x_1 + kx_2 + 4x_3 \end{aligned}$$

is not in \mathbb{R}^3 ? For these value(s) of k find a vector that is not in the range. Also for other values of k find which vector $\mathbf{x}(x_1, x_2, x_3)$ maps into the vector $\mathbf{w}(0, 0, 1)$.

Solution:

The augmented matrix of the given system of equations is

$$\begin{aligned}
& \begin{pmatrix} 1 & 1 & 5 & w_1 \\ 1 & 2 & 7 & w_2 \\ 2 & k & 4 & w_3 \end{pmatrix} && R_{12}(-1), R_{13}(-2) \\
\sim & \begin{pmatrix} 1 & & 1 & 5 & & w_1 \\ 0 & & 1 & 2 & & w_2 - w_1 \\ 0 & k-2 & -6 & w_3 - 2w_1 & & \end{pmatrix} && R_{23}(2-k) \\
\sim & \begin{pmatrix} 1 & 1 & & 5 & & w_1 \\ 0 & 1 & & 2 & & w_2 - w_1 \\ 0 & 0 & -2-2k & w_3 + (2-k)w_2 - (4-k)w_1 & & \end{pmatrix}
\end{aligned}$$

Now two cases arise: (A) $-2 - 2k \neq 0$, and (B) $-2 - 2k = 0$

Case (A) $-2 - 2k \neq 0$, i.e., $k \neq -1$. In this case there is a unique solution for $\mathbf{x}(x_1, x_2, x_3)$ for any vector $\mathbf{w}(w_1, w_2, w_3)$. Thus in this case for each vector $\mathbf{w}(w_1, w_2, w_3)$ there exists a vector $\mathbf{x}(x_1, x_2, x_3)$ for which \mathbf{w} is the image under the given transformation. Specifically if $w = (0, 0, 1)$, then the augmented matrix simplifies to

$$\begin{pmatrix} 1 & 1 & & 5 & 0 \\ 0 & 1 & & 2 & 0 \\ 0 & 0 & -2-2k & 1 & \end{pmatrix}$$

The corresponding system of equations is

$$\begin{aligned}
x_1 + x_2 + 5x_3 &= 0 \\
x_2 + 2x_3 &= 0 \\
(-2 - 2k)x_3 &= 1
\end{aligned}$$

Solving backward we obtain

$$\begin{aligned}
x_3 &= -\frac{1}{2(k+1)}, \quad x_2 = \frac{1}{k+1}, \\
x_1 &= -x_2 - 5x_3 = 2x_3 - 5x_3 = -3x_3 = \frac{3}{2(k+1)}
\end{aligned}$$

Hence if $k \neq -1$, the vector $\left(\frac{3}{2(k+1)}, \frac{1}{k+1}, -\frac{1}{2(k+1)}\right)$ maps into the vector $(0, 0, 1)$ using the given linear operator.

Case (B) $-2 - 2k = 0$ i.e., $k = -1$. For $k = -1$, the augmented matrix takes the form

$$\begin{pmatrix} 1 & 1 & 5 & & w_1 \\ 0 & 1 & 2 & & w_2 - w_1 \\ 0 & 0 & 0 & w_3 + 3w_2 - 5w_1 & \end{pmatrix}$$

Now the discussion centers around the two subcases (B1) $w_3 + 3w_2 - 5w_1 \neq 0$ and (B2) $w_3 + 3w_2 - 5w_1 = 0$.

Case (B1) $w_3 + 3w_2 - 5w_1 \neq 0$. In this case no set of values of (x_1, x_2, x_3) will satisfy the given system, as the last row of the augmented matrix above leads to an inconsistency. Thus when $k = -1$, if (w_1, w_2, w_3) is not on the plane $-5w_1 + 3w_2 + w_3 = 0$, then there is no point (x_1, x_2, x_3) in \mathbb{R}^3 of which (w_1, w_2, w_3) is an image. In other words, the points not lying on this plane in the w -space are not in the range of the linear operator. One such point is $(0, 0, 1)$.

Case (B2) $w_3 + 3w_2 - 5w_1 = 0$. In this case the system of equations becomes consistent, but we can't solve it for x_3 . Therefore we assign an arbitrary value to it, say t . Then we have

$$\begin{aligned} x_3 = t, \quad x_2 + 2x_3 = w_2 - w_1 &\Rightarrow x_2 = w_2 - w_1 - 2t, \\ x_1 + x_2 + 5x_3 = w_1 \\ \Rightarrow x_1 = w_1 - x_2 - 5x_3 = w_1 - (w_2 - w_1 - 2t) - 5t &= 2w_1 - w_2 - 3t \\ \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2w_1 - w_2 \\ w_2 - w_1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \end{aligned}$$

This represents a line in the x -space passing through the point $(2w_1 - w_2, w_2 - w_1, 0)$ with direction vector $(-3, -2, 1)$. The line maps into the point $(w_1, w_2, 5w_1 - 3w_2)$ in the w -space, which suggests that when $k = -1$, the given transformation is a projection mapping.

4. Let V denote the set of ordered triples (x, y, z) and addition be defined on V as in \mathbb{R}^3 . If the following definition is used for scalar multiplication

$$k(x, y, z) = (kx, y, kz)$$

Determine whether V is a vector space. If it is not, state the axioms which are not satisfied.

Solution:

Axioms 1 through 5 will hold because addition is identical to the one in \mathbb{R}^3 . Axiom 6 also holds as $k(x, y, z) = (kx, y, kz) \in \mathbb{R}^3$.

Let $\mathbf{u} = (x, y, z)$, $\mathbf{v} = (x', y', z')$

$$\begin{aligned} \text{Axiom 7: } k(\mathbf{u} + \mathbf{v}) &= k((x, y, z) + (x', y', z')) \\ &= k(x + x', y + y', z + z') \\ &= (k(x + x'), y + y', k(z + z')) \end{aligned}$$

$$\begin{aligned} k\mathbf{u} + k\mathbf{v} &= k(x, y, z) + k(x', y', z') \\ &= (kx, y, kz) + (kx', y', kz') \\ &= (k(x + x'), y + y', k(z + z')) \quad \text{- holds} \end{aligned}$$

$$\text{Axiom 8: } (k + l)\mathbf{u} = (k + l)(x, y, z) = ((k + l)x, y, (k + l)z)$$

$$\begin{aligned} k\mathbf{u} + l\mathbf{u} &= k(x, y, z) + l(x, y, z) \\ &= (kx, y, kz) + (lx, y, lz) \\ &= ((k + l)x, 2y, (k + l)z) \quad \text{- fails} \end{aligned}$$

$$\text{Axiom 9: } k(l\mathbf{u}) = k(l(x, y, z)) = k(lx, y, lz) = (klx, y, klz)$$

$$(kl)\mathbf{u} = (kl)(x, y, z) = (klx, y, klz) \quad \text{- holds}$$

$$\text{Axiom 10: } 1\mathbf{u} = 1(x, y, z) = (1 \times x, y, 1 \times z)$$

$$= (x, y, z) = \mathbf{u} \quad \text{- holds}$$

We note that all axioms hold, except Axiom 8. So V is not a vector space.