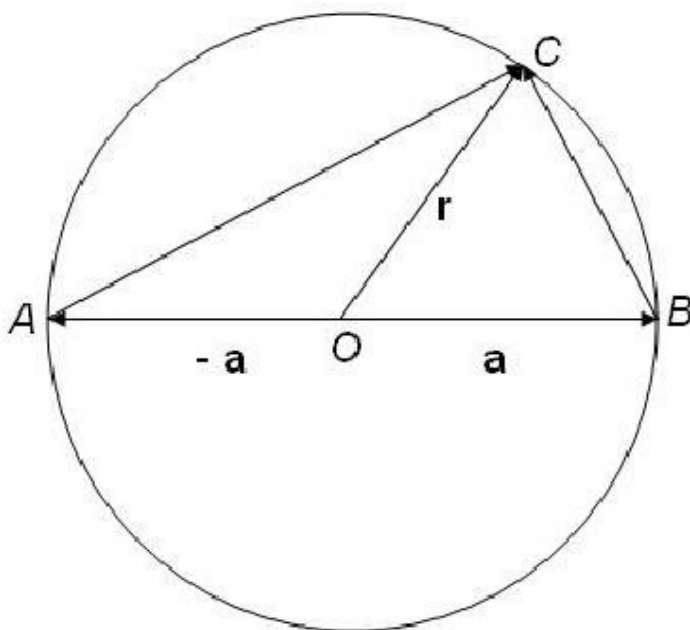


Trinity Western University  
Department of Mathematical Sciences  
MATH 250 (Linear Algebra)  
Sample Mid-Term Exam II Solution

1. Let  $A$  and  $B$  be the end points of a diameter of a circle. If  $C$  is any other point on the circle, show that  $AC$  and  $BC$  are perpendicular.

**Solution:**

Let  $O$  be the center of the circle, and let  $\vec{OB} = \mathbf{a}$ , and  $\vec{OC} = \mathbf{r}$ , then  $\vec{OA} = -\mathbf{a}$   
(See the diagram below)

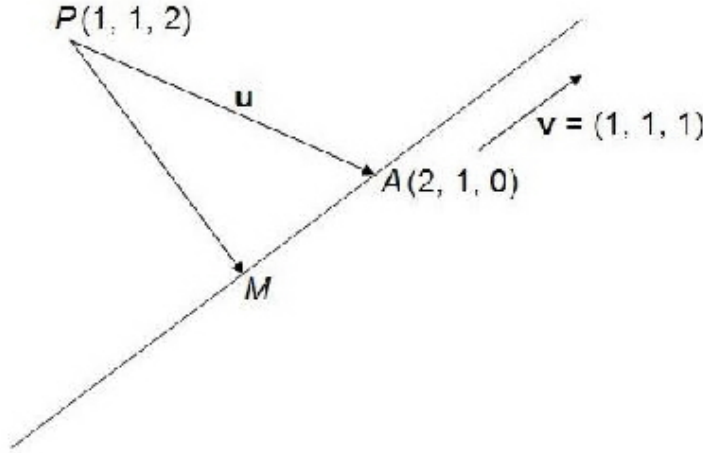


$$\begin{aligned}\vec{BC} &= \mathbf{r} - \mathbf{a}, \quad \vec{AC} = \mathbf{r} - (-\mathbf{a}) = \mathbf{r} + \mathbf{a} \\ \vec{BC} \cdot \vec{AC} &= (\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} + \mathbf{a}) = \mathbf{r} \cdot \mathbf{r} - \mathbf{a} \cdot \mathbf{a} \\ &= |\mathbf{r}|^2 - |\mathbf{a}|^2 \\ &= 0, \text{ since } |\mathbf{r}| = |\mathbf{a}|, \text{ both being the lengths of the radius of the} \\ &\text{same circle.}\end{aligned}$$

2. Find the equation of the line passing through  $P_0(1, 1, 2)$ , intersecting the line  $(x, y, z) = (2, 1, 0) + t(1, 1, 1)$  and perpendicular to that line.

**Solution:**

Let  $M$  be the foot of perpendicular drawn from  $P$  upon the given line (see the diagram below)



$$\mathbf{u} = \overrightarrow{PA} = (2, 1, 0) - (1, 1, 2) = (1, 0, -2)$$

$\overrightarrow{PM}$  is the projection of  $\mathbf{u}$  orthogonal to  $\mathbf{v}$ , the direction vector of the given line.

Therefore

$$\begin{aligned} \overrightarrow{PM} &= \mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u} \\ &= \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} \end{aligned}$$

But  $\mathbf{u} \cdot \mathbf{v} = (1, 0, -2) \cdot (1, 1, 1) = (1)(1) + (0)(1) + (-2)(1) = 1 + 0 - 2 = -1$

and  $|\mathbf{v}|^2 = 1^2 + 1^2 + 1^2 = 1 + 1 + 1 = 3$

$$\begin{aligned} \text{Thus } \overrightarrow{PM} &= (1, 0, -2) - \left(-\frac{1}{3}\right) (1, 1, 1) \\ &= (1, 0, -2) + \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \\ &= \left(\frac{4}{3}, \frac{1}{3}, -\frac{5}{3}\right) \end{aligned}$$

So the direction vector of  $\overrightarrow{PM}$  can be chosen as  $(4, 1, -5)$ . Hence equation of  $PM$  is

$$x = 1 + 4t, \quad y = 1 + t, \quad z = 2 - 5t$$

*Alternately*

For the point  $M$ , there exists some value of  $t$ , such that its coordinates are  $(2 + t, 1 + t, t)$

Therefore  $\overrightarrow{PM} = (2 + t, 1 + t, t) - (1, 1, 2) = (1 + t, t, -2 + t)$

Since  $\overrightarrow{PM} \perp \mathbf{v}$ , we must have  $\overrightarrow{PM} \cdot \mathbf{v} = 0$

$$\Rightarrow (1 + t)(1) + (t)(1) + (-2 + t)(1) = 0$$

$$\Rightarrow 1 + t + t - 2 + t = 0$$

$$\Rightarrow -1 + 3t = 0 \Rightarrow t = \frac{1}{3}$$

Hence  $\overrightarrow{PM} = \left(1 + \frac{1}{3}, \frac{1}{3}, -2 + \frac{1}{3}\right) = \left(\frac{4}{3}, \frac{1}{3}, -\frac{5}{3}\right)$

and we get the equation of the line as before.

3. For what value(s) of  $k$  and  $(w_1, w_2, w_3)$  the range of the linear operator defined by the equations

$$\begin{aligned}w_1 &= x_1 + 2x_2 + x_3 \\w_2 &= -2x_1 + x_2 + 4x_3 \\w_3 &= 7x_1 + 4x_2 + kx_3\end{aligned}$$

is not in  $\mathbb{R}^3$ ?

Also for any value of  $k$ , find which vectors  $(x_1, x_2, x_3)$  map into the line  $w_1 = 1 + 2t, w_2 = 1 + t, w_3 = 1 + 4t$ .

**Solution:**

The augmented matrix is

$$\begin{aligned}&\begin{pmatrix} 1 & 2 & 1 & w_1 \\ -2 & 1 & 4 & w_2 \\ 7 & 4 & k & w_3 \end{pmatrix} && R_{12}(2), R_{13}(-7) \\ \sim &\begin{pmatrix} 1 & 2 & 1 & w_1 \\ 0 & 5 & 6 & w_2 + 2w_1 \\ 0 & -10 & k - 7 & w_3 - 7w_1 \end{pmatrix} && R_{23}(2) \\ \sim &\begin{pmatrix} 1 & 2 & 1 & w_1 \\ 0 & 5 & 6 & w_2 + 2w_1 \\ 0 & 0 & k + 5 & w_3 + 2w_2 - 3w_1 \end{pmatrix}\end{aligned}$$

Clearly if  $k + 5 \neq 0$ , i.e.,  $k \neq -5$ , there will be a unique solution for  $x_1, x_2$ , and  $x_3$  for any vector  $(w_1, w_2, w_3)$ . Thus if  $k \neq -5$ , the range of the linear operator will be  $\mathbb{R}^3$ . But if  $k = -5$ , the third row of the augmented matrix leads to the equation

$$0x_1 + 0x_2 + 0x_3 = w_3 + 2w_2 - 3w_1$$

which has the solution only if  $w_3 + 2w_2 - 3w_1 = 0$ . So if  $k = -5$  and  $w_3 + 2w_2 - 3w_1 \neq 0$ , the linear operator defined by the given equations does not have any image in  $\mathbb{R}^3$ .

When the map is the line  $w_1 = 1 + 2t, w_2 = 1 + t, w_3 = 1 + 4t$ , the augmented matrix becomes

$$\begin{pmatrix} 1 & 2 & 1 & 1 + 2t \\ 0 & 5 & 6 & 3 + 5t \\ 0 & 0 & k + 5 & 0 \end{pmatrix}$$

Now if  $k \neq -5$ , there is a unique solution for  $(x_1, x_2, x_3)$ , namely

$$x_3 = 0, x_2 = \frac{1}{5}(3 + 5t), x_1 = (1 + 2t) - \frac{2}{5}(3 + 5t) = -\frac{1}{5}$$

Hence when  $k \neq -5$ , the line  $(x_1, x_2, x_3) = (-\frac{1}{5}, \frac{3}{5}, 0) + t(0, 1, 0)$  maps into the line  $(w_1, w_2, w_3) = (1, 1, 1) + t(2, 1, 4)$  irrespective of the value of  $k$  (other than  $-5$ ). But if  $k = -5$ , we have the following equations

$$x_1 + 2x_2 + x_3 = 1 + 2t, 5x_2 + 6x_3 = 3 + 5t, 0 = 0$$

$t$  can be readily eliminated between the above two equations by equating the value of  $t$ . We have

$$\begin{aligned}t &= \frac{1}{2}(x_1 + 2x_2 + x_3 - 1) = \frac{1}{5}(5x_2 + 6x_3 - 3) \\ \Rightarrow 5(x_1 + 2x_2 + x_3 - 1) &= 2(5x_2 + 6x_3 - 3) \\ \Rightarrow 5x_1 + 10x_2 + 5x_3 - 5 &= 10x_2 + 12x_3 - 6 \\ \Rightarrow 5x_1 - 7x_3 + 1 &= 0\end{aligned}$$

So when  $k = -5$ , it is the plane  $5x_1 - 7x_3 + 1 = 0$  (and not just the line  $(x_1, x_2, x_3) = (-\frac{1}{5}, \frac{3}{5}, 0) + t(0, 1, 0)$ ) that maps into the given line  $(w_1, w_2, w_3) = (1, 1, 1) + t(2, 1, 4)$ .

4. If  $V$  is a set of ordered pairs  $(x, y)$  of real numbers with the following operations.

$(x, y) + (x', y') = (x + x', y + y' + 1)$  and  $k(x, y) = (kx, ky + k - 1)$ , determine if it is a vector space. If it is not, list all axioms that fail to hold.

**Solution:**

Let, in the following,  $\mathbf{u} = (x, y)$ ,  $\mathbf{v} = (x', y')$ ,  $\mathbf{w} = (x'', y'')$ , and  $k$  and  $l$  be the scalar numbers.

*Axiom 1:* Clearly  $\mathbf{u} + \mathbf{v} = (x + x', y + y' + 1)$  is an ordered pair of real number, and therefore belongs to  $V$ . So Axiom 1 holds.

*Axiom 2:*  $\mathbf{u} + \mathbf{v} = (x + x', y + y' + 1)$ ,  $\mathbf{v} + \mathbf{u} = (x' + x, y' + y + 1) = \mathbf{u} + \mathbf{v}$ , so Axiom 2 holds.

*Axiom 3:*  $\mathbf{u} + \mathbf{v} = (x + x', y + y' + 1)$ ,  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = (x + x', y + y' + 1) + (x'', y'') = (x + x' + x'', y + y' + 1 + y'') = (x + x' + x'', y + y' + y'' + 2)$   
 $\mathbf{v} + \mathbf{w} = (x' + x'', y' + y'' + 1)$ ,  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (x, y) + (x' + x'', y' + y'' + 1) = (x + x' + x'', y + y' + y'' + 1 + 1) = (x + x' + x'', y + y' + y'' + 2)$ ,  
so Axiom 3 holds.

*Axiom 4:* Let  $\mathbf{e} = (z, w)$  be an additive identity, then  $\mathbf{u} + \mathbf{e} = \mathbf{e} + \mathbf{u} = \mathbf{u}$   
 $\Rightarrow (x, y) + (z, w) = (x, y) \Rightarrow (x + z, y + w + 1) = (x, y)$   
 $\Rightarrow x + z = x, y + w + 1 = y \Rightarrow z = 0, w = -1$   
Hence  $(0, -1)$  is an additive identity of  $V$ .

*Axiom 5:* Let  $\mathbf{v}$  be an additive inverse of  $\mathbf{u}$ , then  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} = \mathbf{e}$   
 $\Rightarrow (x, y) + (x', y') = (0, -1) \Rightarrow (x + x', y + y' + 1) = (0, -1)$   
 $\Rightarrow x + x' = 0, y + y' + 1 = -1 \Rightarrow x' = -x, y' = -y - 2$

Hence  $(-x, -y - 2)$  is an additive inverse of  $(x, y)$ , and Axiom 5 holds.

*Axiom 6:* Obviously  $k\mathbf{u} = (kx, ky + k - 1)$  is an ordered pair of real number, and therefore belongs to  $V$ . So Axiom 6 holds.

*Axiom 7:*  $k(\mathbf{u} + \mathbf{v}) = k(x + x', y + y' + 1) = (k(x + x'), k(y + y' + 1) + k - 1) = (k(x + x'), k(y + y' + 2) - 1)$   
 $k\mathbf{u} + k\mathbf{v} = k(x, y) + k(x', y') = (kx, ky + k - 1) + (kx', ky' + k - 1) = (kx + ky, ky + k - 1 + ky' + k - 1 + 1) = (k(x + x'), k(y + y' + 2) - 1)$   
so Axiom 7 holds.

*Axiom 8:*  $(k + l)\mathbf{u} = (k + l)(x, y) = ((k + l)x, (k + l)y + k + l - 1)$   
 $k\mathbf{u} + l\mathbf{u} = k(x, y) + l(x, y) = (kx, ky + k - 1) + (lx, ly + l - 1) = (kx + lx, ky + k - 1 + ly + l - 1 + 1) = ((k + l)x, (k + l)y + k + l - 1)$   
so Axiom 8 holds.

*Axiom 9:*  $k(l\mathbf{u}) = k(l(x, y)) = k(lx, ly + l - 1) = (klx, k(ly + l - 1) + k - 1) = (klx, kl(y + 1) - 1)$

$kl\mathbf{u} = kl(x, y) = (klx, kly + kl - 1) = (klx, kl(y + 1) - 1)$ , so Axiom 9 holds.

*Axiom 10.*  $1\mathbf{u} = 1(x, y) = (x, y + 1 - 1) = (x, y) = \mathbf{u}$ , so Axiom 10 holds.

Since all the ten axioms are satisfied, therefore  $V$  is a vector space.

5. Is the set  $V$  of all  $2 \times 2$  matrices with equal column sums a subspace of  $M_{22}$ ?  
If not, why not?

**Solution:**

Let  $\mathbf{u}, \mathbf{v} \in V$ , and  $k$  be any scalar. We take

$$\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

so that, because of the equal column sum property of  $V$ ,

$$a + c = b + d, \text{ and } p + r = q + s. \quad (*)$$

Now  $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$  clearly belongs to  $V$  ( $0 + 0 = 0 + 0$ ), so  $V$  is not empty.

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} a + p & b + q \\ c + r & d + s \end{pmatrix}$$

But  $a + p + c + r = b + q + d + s$  (On adding the two equations in  $(*)$ ), hence  $\mathbf{u} + \mathbf{v}$  belongs to  $V$ , and the closure property is satisfied for addition.

$$k\mathbf{u} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

Obviously  $ka + kc = kb + kd$  (On multiplying the first equation in  $(*)$  by  $k$ ), hence  $k\mathbf{u}$  belongs to  $V$ , so the closure property is satisfied for scalar multiplication.

Since all the three requirements are satisfied for subspace,  $V$  is a subspace of  $M_{22}$ .