Trinity Western University
Department of Mathematical Sciences
MATH250 (Linear Algebra)
Mid-Term I Examination Solution

1. Discuss the solution of the system of equations

$$\begin{aligned} x - 4y + 2z &= 2 \\ 3x + (a-4)y &= 1 \\ 3x - y + (a-3)z &= 1 \end{aligned}$$

for various values of a (Indicate in each case how many solutions will you get, also giving the solutions if they exist).

Solution:

The augmented matrix is

$$\begin{pmatrix} 1 & -4 & 2 & 2 \\ 3 & a-4 & 0 & 1 \\ 3 & -1 & a-3 & 1 \end{pmatrix} \qquad R_{12}(-3),\ R_{13}(-3)$$

$$\sim \begin{pmatrix} 1 & -4 & 2 & 2 \\ 0 & a+8 & -6 & -5 \\ 0 & 11 & a-9 & -5 \end{pmatrix} \qquad R_{23}$$

$$\sim \begin{pmatrix} 1 & -4 & 2 & 2 \\ 0 & 11 & a-9 & -5 \\ 0 & a+8 & -6 & -5 \end{pmatrix} \qquad R_{23}\left(-\frac{a+8}{11}\right)$$

$$\sim \begin{pmatrix} 1 & -4 & 2 & 2 \\ 0 & 11 & a-9 & -5 \\ 0 & 0 & -6 - \frac{1}{11}(a-9)(a+8) & -5 + \frac{5}{11}(a+8) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -4 & 2 & 2 \\ 0 & 11 & a-9 & -5 \\ 0 & 0 & -\frac{1}{11}(a^2-a-6) & \frac{5}{11}(a-3) \end{pmatrix} \qquad R_3(-11)$$

$$\sim \begin{pmatrix} 1 & -4 & 2 & 2 \\ 0 & 11 & a-9 & -5 \\ 0 & 0 & (a-3)(a+2) & -5(a-3) \end{pmatrix}$$

We look for the first non-zero entry in the third row. It is $(a-3)(a+2)$, unless it is zero. Thus we must make a distinction between the two cases (i) $(a-3)(a+2) \neq 0$, and $(a-3)(a+2) = 0$.

Case (i) $(a-3)(a+2) \neq 0$, i.e., $a \neq 3$ and $a \neq -2$. Peforming $R_3\left(\frac{1}{(a-3)(a+2)}\right)$ on the augmented matrix, we get

$$\begin{pmatrix} 1 & -4 & 2 & 2 \\ 0 & 11 & a-9 & -5 \\ 0 & 0 & 1 & -\dfrac{5}{a+2} \end{pmatrix}$$

The given system is equivalent to

$x - 4y + 2z = 2$
$11y + (a-9)z = -5$
$z = -\dfrac{5}{a+2}$

Performing back-substitution we get

$y = \frac{1}{11}[-5 - (a-9)z] = \dfrac{1}{11}\left[-5 + \dfrac{5(a-9)}{a+2}\right] = -\dfrac{5}{a+2}$

$x = 2 + 4y - 2z = 2 - \dfrac{10}{a+2} = \dfrac{2(a-3)}{a+2}$

So in this case we get the unique solution

$x = \dfrac{2(a-3)}{a+2},\ y = -\dfrac{5}{a+2},\ z = -\dfrac{5}{a+2}$

Case (ii) $(a-3)(a+2) = 0$. Now the augmented matrix becomes

$$\begin{pmatrix} 1 & -4 & 2 & 2 \\ 0 & 11 & a-9 & -5 \\ 0 & 0 & 0 & -5(a-3) \end{pmatrix}$$

Looking for the first non-zero entry, we find that it is $-5(a-3)$, unless it is equal to zero. Therefore, again, we must make a distinction, between the two cases (iia) $a - 3 \neq 0$, and (iib) $a - 3 = 0$.

Case (iia) $a - 3 \neq 0$, and $(a-3)(a+2) = 0$, i.e., $a \neq 3$, and $a = 3$ or $a = -2 \Rightarrow a = -2$ (since $a = 3$ and $a \neq 3$ at the same time is impossible). For this value of a, the augmented matrix takes the form

$$\begin{pmatrix} 1 & -4 & 2 & 2 \\ 0 & 11 & -11 & -5 \\ 0 & 0 & 0 & 25 \end{pmatrix} \qquad R_2\left(\tfrac{1}{11}\right),\ R_3\left(\tfrac{1}{25}\right)$$

$$\sim \begin{pmatrix} 1 & -4 & 2 & 2 \\ 0 & 1 & -1 & -\frac{5}{11} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is the row echelon form of the starting augmented matrix. The last row leads to

$0x + 0y + 0z = 1$

No values of $x,\ y$ and z can satisfy the above equation. Hence when $a = -2$, there is no solution.

Case (iib) $a - 3 = 0$, and $(a-3)(a+2) = 0 \Rightarrow a = 3$. Now the augmented matrix becomes

$$\begin{pmatrix} 1 & -4 & 2 & 2 \\ 0 & 11 & -6 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The equivalent system of equations is

$x - 4y + 2z = 2, 11\ y - 6z = -5,\ 0 = 0$

Solving for the leading variables we obtain

$x = 2 + 4y - 2z,\ y = \frac{1}{11}(-5 + 6z)$

Setting the free variable $z = t$, the following infinitely many solutions are obtained

$x = \frac{2}{11} + \frac{2}{11}t,\ y = -\frac{5}{11} + \frac{6}{11}t,\ z = t,\ t \in \mathbb{R}$

We can summarize the results:

When $a = -2$, there is no solution.

When $a = 3$, there are infinitely many solutions.

Otherwise there is an unique solution.

2. Show that if A and B are 3 x 3 matrices then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Use this fact to show that in general, i.e., for any two n x n matrices A and B, $AB - BA = I_n$ is not possible.

Solution:

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$

Then

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}$$

and

$$BA = \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} + b_{13}a_{31} & b_{11}a_{12} + b_{12}a_{22} + b_{13}a_{32} & b_{11}a_{13} + b_{12}a_{23} + b_{13}a_{33} \\ b_{21}a_{11} + b_{22}a_{21} + b_{23}a_{31} & b_{21}a_{12} + b_{22}a_{22} + b_{23}a_{32} & b_{21}a_{13} + b_{22}a_{23} + b_{23}a_{33} \\ b_{31}a_{11} + b_{32}a_{21} + b_{33}a_{31} & b_{31}a_{12} + b_{32}a_{22} + b_{33}a_{32} & b_{31}a_{13} + b_{32}a_{23} + b_{33}a_{33} \end{pmatrix}$$

Therefore

$\mathrm{tr}(AB) = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33}$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}b_{ji}$$

and

$\mathrm{tr}(BA) = b_{11}a_{11} + b_{12}a_{21} + b_{13}a_{31} + b_{21}a_{12} + b_{22}a_{22} + b_{23}a_{32} + b_{31}a_{13} + b_{32}a_{23} + b_{33}a_{33}$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} b_{ij}a_{ji} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}b_{ji}$$

Clearly

$\mathrm{tr}(AB) = \mathrm{tr}(BA)$

and the result is also valid for $n \times n$ matrices.

For $AB - BA = I_n$ to be true, $\mathrm{tr}(AB - BA) = \mathrm{tr}(I_n)$

But $\mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$, and $\mathrm{tr}(I_n) = n$, and the two cannot be equal. Thus $AB - BA = I_n$ is clearly impossible.

3. Find an LU-decomposition for $A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{pmatrix}$

Solution:

We have

$$A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{pmatrix} \qquad R_{12}(-\tfrac{1}{2}),\ R_{13}(\tfrac{1}{2})$$

$$\sim \begin{pmatrix} 2 & 4 & 2 \\ \boxed{\tfrac{1}{2}} & -1 & 1 \\ \boxed{-\tfrac{1}{2}} & 2 & 3 \end{pmatrix} \qquad R_{23}(2)$$

$$\sim \begin{pmatrix} 2 & 4 & 2 \\ \boxed{\tfrac{1}{2}} & -1 & 1 \\ \boxed{-\tfrac{1}{2}} & \boxed{-2} & 5 \end{pmatrix},$$

the numbers in the boxes being the multipliers.
Hence

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{pmatrix} \text{ and } U = \begin{pmatrix} 2 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

Of course, the LU-decomposition can be done in other ways also, e.g.,

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

and now $L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

4. Compute $\det(A)$ if

$$\det \begin{pmatrix} a & b & c \\ p & q & r \\ x & y & z \end{pmatrix} = 7, \text{ and } A = \begin{pmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{pmatrix}$$

Solution:

We have

$$\det(A) = \begin{vmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{vmatrix}$$

$$= \begin{vmatrix} 2a & 2b & 2c \\ 2p & 2q & 2r \\ 2x & 2y & 2z \end{vmatrix} + \begin{vmatrix} 2a & 2b & 2c \\ 2p & 2q & 2r \\ a & b & c \end{vmatrix} + \begin{vmatrix} 2a & 2b & 2c \\ x & y & z \\ 2x & 2y & 2z \end{vmatrix} + \begin{vmatrix} 2a & 2b & 2c \\ x & y & z \\ a & b & c \end{vmatrix} +$$

$$+ \begin{vmatrix} p & q & r \\ 2p & 2q & 2r \\ 2x & 2y & 2z \end{vmatrix} + \begin{vmatrix} p & q & r \\ 2p & 2q & 2r \\ a & b & c \end{vmatrix} + \begin{vmatrix} p & q & r \\ x & y & z \\ 2x & 2y & 2z \end{vmatrix} + \begin{vmatrix} p & q & r \\ x & y & z \\ a & b & c \end{vmatrix}$$

Except for the first and the last, all other determinants become zero, because they have proportional rows.

Hence

$$\det(A) = \begin{vmatrix} 2a & 2b & 2c \\ 2p & 2q & 2r \\ 2x & 2y & 2z \end{vmatrix} + \begin{vmatrix} p & q & r \\ x & y & z \\ a & b & c \end{vmatrix}$$

$$= 8 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} - \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}, \text{ on interchanging the first and third rows}$$

$$= 8 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}, \text{ on interchanging the second and third rows}$$

$$= 9 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 9 \times 7 = 63.$$

Alternately we have

$$A = \begin{pmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} a & b & c \\ p & q & r \\ x & y & z \end{pmatrix}$$

$$\Rightarrow \det(A) = \det \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} \det \begin{pmatrix} a & b & c \\ p & q & r \\ x & y & z \end{pmatrix}$$

But $\det \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} = 2 \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = 2(4) - (-1) = 9$

and $\det \begin{pmatrix} a & b & c \\ p & q & r \\ x & y & z \end{pmatrix} = 7$ (given)

Hence $\det(A) = 9 \times 7 = 63$

We can also solve it as under.

We have

$$\det(A) = \begin{vmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{vmatrix} \qquad R_{12}(-2)$$

$$= \begin{vmatrix} 2a+p & 2b+q & 2c+r \\ x-4a & y-4b & z-4c \\ 2x+a & 2y+b & 2z+c \end{vmatrix} \qquad R_{23}(-2)$$

$$= \begin{vmatrix} 2a+p & 2b+q & 2c+r \\ x-4a & y-4b & z-4c \\ 9a & 9b & 9c \end{vmatrix}$$

$$= 9\begin{vmatrix} 2a+p & 2b+q & 2c+r \\ x-4a & y-4b & z-4c \\ a & b & c \end{vmatrix} \qquad R_{31}(-2),\ R_{32}(4)$$

$$= 9\begin{vmatrix} p & q & r \\ x & y & z \\ a & b & c \end{vmatrix} = 9\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 9 \times 7 = 63$$

5. A lab rat has a choice of three foods P, Q and R each day. On any given day it has a 60% chance of choosing the same food as it chose the previous day, and is equally likely to choose either of the other foods.
 a) If it chooses P one day, find the probability it chooses Q three days later.
 b) What percentage of its meals are food P, Q and R?

Solution:

The transition matrix is

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}$$

(a) We have $\mathbf{x}^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, then

$$\mathbf{x}^{(1)} = P\mathbf{x}^{(0)} = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.2 \\ 0.2 \end{pmatrix}$$

$$\mathbf{x}^{(2)} = P\mathbf{x}^{(1)} = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.2 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.44 \\ 0.28 \\ 0.28 \end{pmatrix}$$

$$\mathbf{x}^{(3)} = P\mathbf{x}^{(2)} = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix} \begin{pmatrix} 0.44 \\ 0.28 \\ 0.28 \end{pmatrix} = \begin{pmatrix} 0.376 \\ 0.312 \\ 0.312 \end{pmatrix}$$

Thus the probability of the rat choosing Q (or R) three days later is 0.312.

(b) Let the steady state be $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$

We have

$$P\mathbf{q} = \mathbf{q} \Rightarrow \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

$\Rightarrow 0.6q_1 + 0.2q_2 + 0.2q_3 = q_1,\ 0.2q_1 + 0.6q_2 + 0.2q_3 = q_2,\ 0.2q_1 + 0.2q_2 + 0.6q_3 = q_3.$

$\Rightarrow -0.4q_1 + 0.2q_2 + 0.2q_3 = 0,\ 0.2q_1 - 0.4q_2 + 0.2q_3 = 0,\ 0.2q_1 + 0.2q_2 - 0.4q_3 = 0.$

$\Rightarrow -2q_1 + q_2 + q_3 = 0,\ q_1 - 2q_2 + q_3 = 0,\ q_1 + q_2 - 2q_3 = 0$
Solving we get

$$\frac{q_1}{1} = \frac{q_2}{1} = \frac{q_3}{1} = \frac{q_1 + q_2 + q_3}{3} = \frac{1}{3}$$

$\Rightarrow q_1 = q_2 = q_3 = \frac{1}{3}$,
a result to be expected, as the rat does not show any preference for any of the three types of food.