Trinity Western University
Department of Mathematical Sciences
MATH 250 (Linear Algebra)
Sample Mid-Term Exam I Solution

1. Show that the matrix

$$A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix}$$

satisfies the equation

$$(A+3I)^2(A-I) = 0$$

Use the above equation to prove that A is invertible and compute A^{-1}.

Solution:

We have

$$A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix}$$

$$\Rightarrow A+3I = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 8 & 16 \\ 4 & 4 & 8 \\ -4 & -4 & -8 \end{pmatrix}$$

$$\Rightarrow (A+3I)^2 = \begin{pmatrix} 8 & 8 & 16 \\ 4 & 4 & 8 \\ -4 & -4 & -8 \end{pmatrix} \begin{pmatrix} 8 & 8 & 16 \\ 4 & 4 & 8 \\ -4 & -4 & -8 \end{pmatrix} = \begin{pmatrix} 32 & 32 & 64 \\ 16 & 16 & 32 \\ -16 & -16 & -32 \end{pmatrix}$$

$$A-I = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 16 \\ 4 & 0 & 8 \\ -4 & -4 & -12 \end{pmatrix}$$

$$\Rightarrow (A+3I)^2(A-I)$$

$$= \begin{pmatrix} 32 & 32 & 64 \\ 16 & 16 & 32 \\ -16 & -16 & -32 \end{pmatrix} \begin{pmatrix} 4 & 8 & 16 \\ 4 & 0 & 8 \\ -4 & -4 & -12 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Expanding we get

$$(A^2+6A+9I)(A-I) = 0$$
$$\Rightarrow A^3+5A^2+3A-9I = 0$$
$$\Rightarrow A^3+5A^2+3A = 9I$$
$$\Rightarrow A(A^2+5A+3I) = 9I$$
$$\Rightarrow A[\tfrac{1}{9}(A^2+5A+3I)] = I$$

If we denote by B the matrix $\frac{1}{9}(A^2+5A+3I)$, then

$$AB = I$$

which shows that A is invertible and $A^{-1} = B = \frac{1}{9}(A^2+5A+3I)$

Now

$$A^2+5A+3I = (A+3I)^2 - (A+3I) - 3I$$

$$= \begin{pmatrix} 32 & 32 & 64 \\ 16 & 16 & 32 \\ -16 & -16 & -32 \end{pmatrix} - \begin{pmatrix} 8 & 8 & 16 \\ 4 & 4 & 8 \\ -4 & -4 & -8 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 21 & 24 & 48 \\ 12 & 9 & 24 \\ -12 & -12 & -27 \end{pmatrix}$$

Hence

$$A^{-1} = \tfrac{1}{9}(A^2 + 5A + 3I) = \frac{1}{9} \begin{pmatrix} 21 & 24 & 48 \\ 12 & 9 & 24 \\ -12 & -12 & -27 \end{pmatrix} = \begin{pmatrix} \frac{7}{3} & \frac{8}{3} & \frac{16}{3} \\ \frac{4}{3} & 1 & \frac{8}{3} \\ -\frac{4}{3} & -\frac{4}{3} & -3 \end{pmatrix}$$

2. Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 5 \\ 1 & -7 & 13 \end{pmatrix}$$

Show that A is not invertible by finding a lower-triangular matrix L such that $A = LU$, where U is an upper-triangular matrix which has at least one row of zeros.

Solution:

We have

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 5 \\ 1 & -7 & 13 \end{pmatrix} \qquad R_{12}(-2),\ R_{13}(-1)$$

$$\sim \begin{pmatrix} 1 & 3 & -1 \\ \boxed{2} & -5 & 7 \\ \boxed{1} & -10 & 14 \end{pmatrix} \qquad R_{23}(-2)$$

$$\sim \begin{pmatrix} 1 & 3 & -1 \\ \boxed{2} & -5 & 7 \\ \boxed{1} & \boxed{2} & 0 \end{pmatrix}$$

Thus we have

$$A = LU$$

where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \text{ and } U = \begin{pmatrix} 1 & 3 & -1 \\ 0 & -5 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

Clearly A is not invertible, because of its row-echelon form, it is row equivalent of U, which is not invertible as it has got a row of zeros.

3. Find for what values of c the following matrix is not invertible. Find the inverse of the matrix for the remaining values of c.

$$\begin{pmatrix} 1 & 0 & -c \\ -1 & 3 & 1 \\ 0 & 2c & -4 \end{pmatrix}$$

Solution:

Let $A = \begin{pmatrix} 1 & 0 & -c \\ -1 & 3 & 1 \\ 0 & 2c & -4 \end{pmatrix}$

Obviously A is not invertible if $\det(A) = 0$

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 1 & 0 & -c \\ -1 & 3 & 1 \\ 0 & 2c & -4 \end{vmatrix} \qquad R_{12}(1) \\
&= \begin{vmatrix} 1 & 0 & -c \\ 0 & 3 & 1-c \\ 0 & 2c & -4 \end{vmatrix} \qquad R_{23}(-2c/3) \\
&= \begin{vmatrix} 1 & 0 & -c \\ 0 & 3 & 1-c \\ 0 & 0 & -4-\frac{2}{3}c(1-c) \end{vmatrix} \\
&= (1)(3)(-4-\tfrac{2}{3}c(1-c)) \\
&= -12-2c+2c^2 = 2(c^2-c-6) \\
&= 2(c-3)(c+2)
\end{aligned}$$

Thus A is not invertible if $c = 3$ or $c = -2$.

For other values of c, we have

$$\operatorname{Adj}(A) = \begin{pmatrix} -12-2c & -2c^2 & 3c \\ -4 & -4 & -1+c \\ -2c & -2c & 3 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)}\operatorname{Adj}(A) = \frac{1}{2(c-3)(c+2)} \begin{pmatrix} -12-2c & -2c^2 & 3c \\ -4 & -4 & -1+c \\ -2c & -2c & 3 \end{pmatrix}$$

4. Using Cramer's rule solve the following system of equations for z:

$$\begin{aligned}
x+y+z+w &= 10 \\
x+2y+3z+4w &= 30 \\
x+4y+9z+16w &= 100 \\
x+8y+27z+64w &= 354
\end{aligned}$$

Solution:

We have

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 & 10 & 1 \\ 1 & 2 & 30 & 4 \\ 1 & 4 & 100 & 16 \\ 1 & 8 & 354 & 64 \end{pmatrix}$$

$$z = \frac{\det(A_3)}{\det(A)} = \begin{vmatrix} 1 & 1 & 10 & 1 \\ 1 & 2 & 30 & 4 \\ 1 & 4 & 100 & 16 \\ 1 & 8 & 354 & 64 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 10 \\ 1 & 2 & 4 & 30 \\ 1 & 4 & 16 & 100 \\ 1 & 8 & 64 & 354 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 3 \\ 1 & 4 & 16 & 9 \\ 1 & 8 & 64 & 27 \end{vmatrix}$$

We shall be calculating the values of the two determinants simultaneously using the elementary row operations. First we form an augmented "determinant"

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 10 \\ 1 & 2 & 4 & 3 & 30 \\ 1 & 4 & 16 & 9 & 100 \\ 1 & 8 & 64 & 27 & 354 \end{vmatrix} \qquad R_{12}(-1),\ R_{13}(-1),\ R_{14}(-1)$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 & 10 \\ 0 & 1 & 3 & 2 & 20 \\ 0 & 3 & 15 & 8 & 90 \\ 0 & 7 & 63 & 26 & 344 \end{vmatrix} \qquad R_{23}(-3),\ R_{24}(-7)$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 & 10 \\ 0 & 1 & 3 & 2 & 20 \\ 0 & 0 & 6 & 2 & 30 \\ 0 & 0 & 42 & 12 & 204 \end{vmatrix} \qquad R_{34}(-7)$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 & 10 \\ 0 & 1 & 3 & 2 & 20 \\ 0 & 0 & 6 & 2 & 30 \\ 0 & 0 & 0 & -2 & -6 \end{vmatrix}$$

Hence $z = \dfrac{(1)(1)(6)(-6)}{(1)(1)(6)(-2)} = 3$

5. Assume that there are three classes – upper U, middle M, and lower L – and that social mobility is modeled as follows:

i) Of children of U parents, 70% remain U while 20% become L and 10% become M.

ii) Of children of M parents, 80% remain M while 10% become L and 10% become U.

iii) Of children of L parents, 60% remain L while 10% become U and 30% become M.

Find the probability that the grandchild of L parents becomes U. Also find the long-term breakdown of society into classes.

Solution:

The transition matrix is

$$P = \begin{pmatrix} 0.7 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.3 \\ 0.2 & 0.1 & 0.6 \end{pmatrix}$$

If a person belongs to L at a specific time, the state vector at that time is $\mathbf{x}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

The state vector for the next generation is

$$\mathbf{x}^{(1)} = P\mathbf{x}^{(0)} = \begin{pmatrix} 0.7 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.3 \\ 0.2 & 0.1 & 0.6 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.3 \\ 0.6 \end{pmatrix},$$

and for the further next generation is

$$\mathbf{x}^{(2)} = P\mathbf{x}^{(1)} = \begin{pmatrix} 0.7 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.3 \\ 0.2 & 0.1 & 0.6 \end{pmatrix} \begin{pmatrix} 0.1 \\ 0.3 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.16 \\ 0.43 \\ 0.41 \end{pmatrix}$$

Hence the probability of a grandchild of L person becoming U is 16%.

Let $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$ be the steady state vector. We have

$P\mathbf{q} = \mathbf{q}$

$\Rightarrow \begin{pmatrix} 0.7 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.3 \\ 0.2 & 0.1 & 0.6 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$

$\Rightarrow 0.7q_1 + 0.1q_2 + 0.1q_3 = q_1,\ 0.1q_1 + 0.8q_2 + 0.3q_3 = q_2,\ 0.2q_1 + 0.1q_2 + 0.6q_3 = q_3$

$\Rightarrow -0.3q_1 + 0.1q_2 + 0.1q_3 = 0,\ 0.1q_1 - 0.2q_2 + 0.3q_3 = 0,\ 0.2q_1 + 0.1q_2 - 0.4q_3 = 0$

$\Rightarrow -3q_1 + q_2 + q_3 = 0,\ q_1 - 2q_2 + 3q_3 = 0,\ 2q_1 + q_2 - 4q_3 = 0$

The augmented matrix for the above system is

$$\begin{pmatrix} -3 & 1 & 1 & 0 \\ 1 & -2 & 3 & 0 \\ 2 & 1 & -4 & 0 \end{pmatrix} \qquad R_{12}(\tfrac{1}{3}),\ R_{13}(\tfrac{2}{3})$$

$$\sim \begin{pmatrix} -3 & 1 & 1 & 0 \\ 0 & -\frac{5}{3} & \frac{10}{3} & 0 \\ 0 & \frac{5}{3} & -\frac{10}{3} & 0 \end{pmatrix} \qquad R_{23}(1)$$

$$\sim \begin{pmatrix} -3 & 1 & 1 & 0 \\ 0 & -\frac{5}{3} & \frac{10}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The equivalent system of equations is

$$-3q_1 + q_2 + q_3 = 0,\ \ -\tfrac{5}{3}q_2 + \tfrac{10}{3}q_3 = 0,\ 0q_3 = 0$$

Since the last equation is satisfied by any value of q_3, we set $q_3 = t$.

Solving backward we obtain

$$q_2 = 2q_3 = 2t,\ q_1 = \tfrac{1}{3}(q_2 + q_3) = \tfrac{1}{3}(2t + t) = t$$

Hence we have the general solution

$$q_1 = t,\ q_2 = 2t,\ q_3 = t$$

But $q_1,\ q_2$ and q_3 satisfy the equation

$$q_1 + q_2 + q_3 = 1 \Rightarrow t + 2t + t = 1 \Rightarrow 4t = 1 \Rightarrow t = \tfrac{1}{4}$$

$$\Rightarrow q_1 = \tfrac{1}{4},\ q_2 = \tfrac{1}{2},\ q_3 = \tfrac{1}{4}$$

Thus in the long-term 50% of the people will belong to the middle class, while the remaining 50% will be equally divided into the upper and the lower classes.