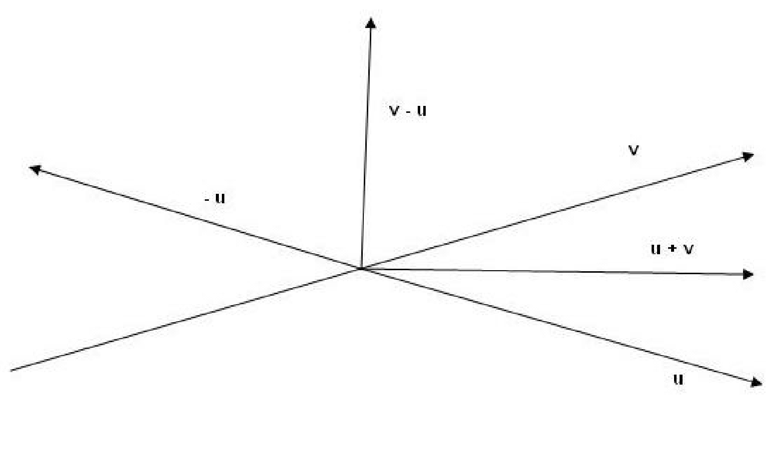
Trinity Western University
Department of Mathematical Sciences
MATH250 (Linear Algebra)
Mid-Term II Examination Solution

1. Show that if $\mathbf{u}$ and $\mathbf{v}$ are **unit** vectors then the vector $\mathbf{u} + \mathbf{v}$ bisects the direction of the vectors $\mathbf{u}$ and $\mathbf{v}$.
Use the above fact to find the equations of the lines bisecting the two intersecting lines
L_1: $(x, y, z) = (-1, 0, 2) + t(2, 2, 1)$ and $L_2 : (x, y, z) = (-1, 0, 2) + t(6, -2, 3)$

Solution:

Let $\mathbf{u} + \mathbf{v}$ make angles α and β with $\mathbf{u}$ and $\mathbf{v}$ respectively.
We have
$$\cos\alpha = \frac{(\mathbf{u}+\mathbf{v})\cdot\mathbf{u}}{|\mathbf{u}+\mathbf{v}|\ |\mathbf{u}|} = \frac{\mathbf{u}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{u}}{|\mathbf{u}+\mathbf{v}|\ |\mathbf{u}|} = \frac{1+\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}+\mathbf{v}|}, \text{ since } \mathbf{u} \text{ is a unit vector,}$$
$$\cos\beta = \frac{(\mathbf{u}+\mathbf{v})\cdot\mathbf{v}}{|\mathbf{u}+\mathbf{v}|\ |\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v}}{|\mathbf{u}+\mathbf{v}|\ |\mathbf{v}|} = \frac{1+\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}+\mathbf{v}|}, \text{ since } \mathbf{v} \text{ is a unit vector.}$$
Since $\cos\alpha = \cos\beta$, it follows that $\alpha = \beta$, and $\mathbf{u} + \mathbf{v}$ bisects the direction of the vectors $\mathbf{u}$ and $\mathbf{v}$.

The direction vectors along the two lines are $\mathbf{u} = (2, 2, 1)$, and $\mathbf{v} = (6, -2, 3)$. If we wish to use the above fact, we require these vectors to be **unit** vectors (i.e., of unit length). They can be readily converted to unit vectors by dividing them by their respective lengths. Thus we have
$$\widehat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{\sqrt{2^2+2^2+1^2}}(2, 2, 1) = \tfrac{1}{3}(2, 2, 1) = (\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3})$$
$$\widehat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{6^2+(-2)^2+3^2}}(6, -2, 3) = \tfrac{1}{7}(6, -2, 3) = (\tfrac{6}{7}, -\tfrac{2}{7}, \tfrac{3}{7})$$
The two bisectors have the direction vectors $\widehat{\mathbf{u}} + \widehat{\mathbf{v}}$ and $\widehat{\mathbf{v}} - \widehat{\mathbf{u}}$ (see the diagram below).

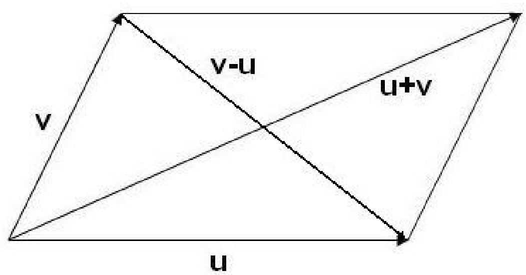


$\widehat{\mathbf{u}} + \widehat{\mathbf{v}} = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3}) + (\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}) = (\frac{32}{21}, \frac{8}{21}, \frac{16}{21})$
$\widehat{\mathbf{v}} - \widehat{\mathbf{u}} = (\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}) - (\frac{2}{3}, \frac{2}{3}, \frac{1}{3}) = (\frac{4}{21}, -\frac{20}{21}, \frac{2}{21})$
There is no loss of generality in taking these vectors as (4, 1, 2) and (2, -10, 1) respectively.
Hence the two bisectors are
$(x, y, z) = (-1, 0, 2) + t(4, 1, 2)$ and $(x, y, z) = (-1, 0, 2) + t(2, -10, 1)$

2. Show that if the diagonals of a parallelogram are perpendicular, it is necessarily a rhombus.

Solution:

Let the two adjacent sides of the parallelogram be $\mathbf{u}$ and $\mathbf{v}$. Then the two diagonals are $\mathbf{u} + \mathbf{v}$ and $\mathbf{v} - \mathbf{u}$. (See diagram below)



If they are perpendicular to each other, we must have
$(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{v} - \mathbf{u}) = 0$
$\Rightarrow \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{u} = 0$
$\Rightarrow |\mathbf{v}|^2 - |\mathbf{u}|^2 = 0$
$\Rightarrow |\mathbf{v}|^2 = |\mathbf{u}|^2$
$\Rightarrow |\mathbf{v}| = |\mathbf{u}|$
Since the lengths of the adjacent sides of the parallelogram are equal, it must be a rhombus.

3. For what value(s) of k the range of the linear operator defined by the equations
$w_1 = x_1 + x_2 + 5x_3$
$w_2 = x_1 + 2x_2 + 7x_3$
$w_3 = 2x_1 + kx_2 + 4x_3$
is not in $\mathbb{R}^3$? For these value(s) of k find a vector that is not in the range. Also for other values of k find which vector $\mathbf{x}(x_1,\ x_2,\ x_3)$ maps into the vector $\mathbf{w}(0,\ 0,\ 1)$..

Solution:

The augmented matrix of the given system of equations is

$$\begin{pmatrix} 1 & 1 & 5 & w_1 \\ 1 & 2 & 7 & w_2 \\ 2 & k & 4 & w_3 \end{pmatrix} \qquad R_{12}(-1),\ R_{13}(-2)$$

$$\sim \begin{pmatrix} 1 & 1 & 5 & w_1 \\ 0 & 1 & 2 & w_2 - w_1 \\ 0 & k-2 & -6 & w_3 - 2w_1 \end{pmatrix} \qquad R_{23}(2-k)$$

$$\sim \begin{pmatrix} 1 & 1 & 5 & w_1 \\ 0 & 1 & 2 & w_2 - w_1 \\ 0 & 0 & -2-2k & w_3 + (2-k)w_2 - (4-k)w_1 \end{pmatrix}$$

Now two cases arise: (A) $-2-2k \neq 0$, and (B) $-2-2k = 0$

Case (A) $-2-2k \neq 0$, i.e., $k \neq -1$. In this case there is a unique solution for $\mathbf{x}(x_1, x_2, x_3)$ for any vector $\mathbf{w}(w_1,\ w_2,\ w_3)$. Thus in this case for each vector $\mathbf{w}(w_1,\ w_2,\ w_3)$ there exists a vector $\mathbf{x}(x_1, x_2, x_3)$ for which $\mathbf{w}$ is the image under the given transformation. Specifically if $w = (0,\ 0,\ 1)$, then the augmented matrix simplifies to

$$\begin{pmatrix} 1 & 1 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2-2k & 1 \end{pmatrix}$$

The corresponding system of equations is

$$x_1 + x_2 + 5x_3 = 0$$
$$x_2 + 2x_3 = 0$$
$$(-2-2k)x_3 = 1$$

Solving backword we obtain

$$x_3 = -\frac{1}{2(k+1)},\ x_2 = \frac{1}{k+1},$$

$$x_1 = -x_2 - 5x_3 = 2x_3 - 5x_3 = -3x_3 = \frac{3}{2(k+1)}$$

Hence if $k \neq 1$, the vector $\left(\frac{3}{2(k+1)},\ \frac{1}{k+1},\ -\frac{1}{2(k+1)}\right)$ maps into the vector (0, 0, 1) using the given linear operator.

Case (B) $-2-2k = 0$ i.e., $k = -1$. For $k = -1$, the augmented matrix takes the form

$$\begin{pmatrix} 1 & 1 & 5 & w_1 \\ 0 & 1 & 2 & w_2 - w_1 \\ 0 & 0 & 0 & w_3 + 3w_2 - 5w_1 \end{pmatrix}$$

Now the discussion centers around the two subcases (B1) $w_3 + 3w_2 - 5w_1 \neq 0$ and (B2) $w_3 + 3w_2 - 5w_1 = 0$.

Case (B1) $w_3 + 3w_2 - 5w_1 \neq 0$. In this case no set of values of (x_1, x_2, x_3) will satisfy the given system, as the last row of the augmented matrix above leads to an inconsistency. Thus when $k = -1$, if (w_1, w_2, w_3) is not on the plane $-5w_1 + 3w_2 + w_3 = 0$, then there is no point (x_1, x_2, x_3) in $\mathbb{R}^3$ of which (w_1, w_2, w_3) is an image. In other words, the points not lying on this plane in the w-space are not in the range of the linear operator. One such point is (0, 0, 1).

Case (B2) $w_3 + 3w_2 - 5w_1 = 0$. In this case the system of equations becomes consistent, but we can't solve it for x_3. Therefore we assign an arbitrary value to it, say t. Then we have

$x_3 = t,\ x_2 + 2x_3 = w_2 - w_1 \Rightarrow x_2 = w_2 - w_1 - 2t,$

$x_1 + x_2 + 5x_3 = w_1$

$\Rightarrow x_1 = w_1 - x_2 - 5x_3 = w_1 - (w_2 - w_1 - 2t) - 5t = 2w_1 - w_2 - 3t$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2w_1 - w_2 \\ w_2 - w_1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$$

This represents a line in the x-space passing through the point $(2w_1 - w_2, w_2 - w_1,\ 0)$ with direction vector (-3, -2, 1). The line maps into the point $(w_1,\ w_2,\ 5w_1 - 3w_2)$ in the w-space, which suggests that when $k = -1$, the given transformation is a projection mapping.

4. Let V denote the set of ordered triples (x, y, z) and addition be defined on V as in $\mathbb{R}^3$. If the following definition is used for scalar multiplication

$k(x, y, z) = (kx, y, kz)$

Determine whether V is a vector space. If it is not, state the axioms which are not satisfied.

Solution:

Axioms 1 through 5 will hold because addition is identical to the one in $\mathbb{R}^3$. Axiom 6 also holds as $k(x, y, z) = (kx, y, kz) \in \mathbb{R}^3$.

Let $\mathbf{u} = (x,\ y,\ z),\ \mathbf{v} = (x',\ y',\ z')$

Axiom 7: $k(\mathbf{u} + \mathbf{v}) = k((x,\ y,\ z) + (x',\ y',\ z'))$
$= k(x + x',\ y + y',\ z + z')$
$= (k(x + x'),\ y + y',\ k(z + z'))$

$k\mathbf{u} + k\mathbf{v} = k(x,\ y,\ z) + k(x',\ y',\ z')$
$= (kx,\ y,\ kz) + (kx',\ y',\ kz')$
$= (k(x + x'),\ y + y',\ k(z + z'))$ - holds

Axiom 8: $(k + l)\mathbf{u} = (k + l)(x,\ y,\ z) = ((k + l)x,\ y,\ (k + l)z)$

$k\mathbf{u} + l\mathbf{u} = k(x,\ y,\ z) + l(x,\ y,\ z)$
$= (kx,\ y,\ kz) + (lx,\ y,\ lz)$
$= ((k + l)x,\ 2y,\ (k + l)z)$ - fails

Axiom 9: $k(l\mathbf{u}) = k(l(x,\ y,\ z)) = k(lx,\ y,\ lz) = (klx,\ y,\ klz)$

$(kl)\mathbf{u} = (kl)(x,\ y,\ z) = (klx,\ y,\ klz)$ - holds

Axiom 10: $1\mathbf{u} = 1(x,\ y,\ z) = (1 \times x,\ y,\ 1 \times z)$
$= (x,\ y,\ z) = \mathbf{u}$ - holds

We note that all axioms hold, except Axiom 8. So V is not a vector space.