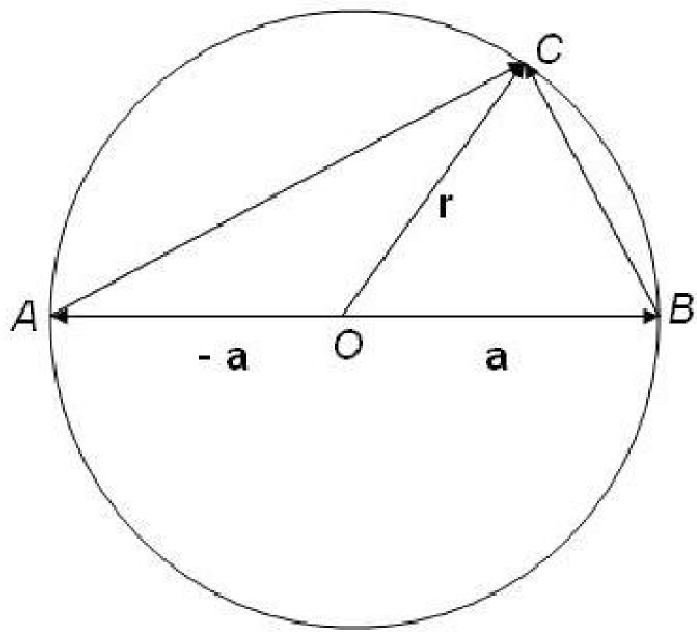
Trinity Western University
Department of Mathematical Sciences
MATH 250 (Linear Algebra)
Sample Mid-Term Exam II Solution

1. Let A and B be the end points of a diameter of a circle. If C is any other point on the circle, show that AC and BC are perpendicular.

Solution:

Let O be the center of the circle, and let $\overrightarrow{OB} = \mathbf{a}$, and $\overrightarrow{OC} = \mathbf{r}$, then $\overrightarrow{OA} = -\mathbf{a}$ (See the diagram below)

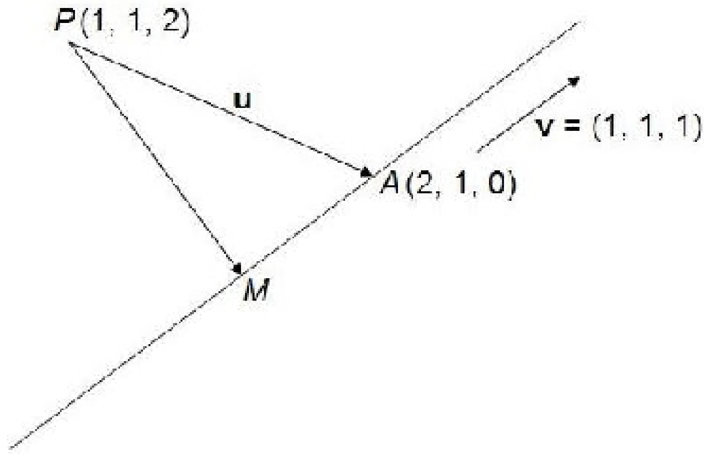


$\overrightarrow{BC} = \mathbf{r} - \mathbf{a}$, $\overrightarrow{AC} = \mathbf{r} - (-\mathbf{a}) = \mathbf{r} + \mathbf{a}$
$\overrightarrow{BC} \cdot \overrightarrow{AC} = (\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} + \mathbf{a}) = \mathbf{r} \cdot \mathbf{r} - \mathbf{a} \cdot \mathbf{a}$
$= |\mathbf{r}|^2 - |\mathbf{a}|^2$
$= 0$, since $|\mathbf{r}| = |\mathbf{a}|$, both being the lengths of the radius of the same circle.

2. Find the equation of the line passing though $P_0(1,\ 1,\ 2)$, intersecting the line $(x, y, z) = (2, 1, 0) + t(1, 1, 1)$ and perpendicular to that line.

Solution:

Let M be the foot of perpendicular drawn from P upon the given line (see the diagram below)



$\mathbf{u} = \overrightarrow{PA} = (2,1,0) - (1,1,2) = (1,0,-2)$

$\overrightarrow{PM}$ is the projection of $\mathbf{u}$ orthogonal to $\mathbf{v}$, the direction vector of the given line.

Therefore

$$\overrightarrow{PM} = \mathbf{u} - \text{proj}_{\mathbf{v}}\, \mathbf{u}$$
$$= \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}$$

But $\mathbf{u}\cdot\mathbf{v} = (1,0,-2)\cdot(1,1,1) = (1)(1)+(0)(1)+(-2)(1) = 1+0-2 = -1$

and $|\mathbf{v}|^2 = 1^2+1^2+1^2 = 1+1+1 = 3$

Thus $\overrightarrow{PM} = (1,0,-2) - \left(-\frac{1}{3}\right)(1,1,1)$

$$= (1,0,-2) + \left(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}\right)$$
$$= \left(\tfrac{4}{3},\tfrac{1}{3},-\tfrac{5}{3}\right)$$

So the direction vector of $\overrightarrow{PM}$ can be chosen as (4, 1, -5). Hence equation of PM is

$$x = 1+4t,\ y = 1+t,\ z = 2-5t$$

Alternately

For the point M, there exists some value of t, such that its coordinates are $(2+t, 1+t, t)$

Therefore $\overrightarrow{PM} = (2+t,1+t,t) - (1,1,2) = (1+t, t, -2+t)$

Since $\overrightarrow{PM} \perp \mathbf{v}$, we must have $\overrightarrow{PM}\cdot\mathbf{v} = 0$

$$\Rightarrow (1+t)(1)+(t)(1)+(-2+t)(1) = 0$$
$$\Rightarrow 1+t+t-2+t = 0$$
$$\Rightarrow -1+3t = 0 \Rightarrow t = \tfrac{1}{3}$$

Hence $\overrightarrow{PM} = \left(1+\frac{1}{3},\frac{1}{3},-2+\frac{1}{3}\right) = \left(\frac{4}{3},\frac{1}{3},-\frac{5}{3}\right)$

and we get the equation of the line as before.

3. For what value(s) of k and $(w_1,\ w_2,\ w_3)$ the range of the linear operator defined by the equations

$$w_1 = x_1 + 2x_2 + x_3$$
$$w_2 = -2x_1 + x_2 + 4x_3$$
$$w_3 = 7x_1 + 4x_2 + kx_3$$

is not in $\mathbb{R}^3$?
Also for any value of k, find which vectors (x_1, x_2, x_3) map into the line $w_1 = 1 + 2t, w_2 = 1 + t, w_3 = 1 + 4t$.

Solution:

The augmented matrix is

$$\begin{pmatrix} 1 & 2 & 1 & w_1 \\ -2 & 1 & 4 & w_2 \\ 7 & 4 & k & w_3 \end{pmatrix} \qquad R_{12}(2),\ R_{13}(-7)$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & w_1 \\ 0 & 5 & 6 & w_2 + 2w_1 \\ 0 & -10 & k-7 & w_3 - 7w_1 \end{pmatrix} \qquad R_{23}(2)$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & w_1 \\ 0 & 5 & 6 & w_2 + 2w_1 \\ 0 & 0 & k+5 & w_3 + 2w_2 - 3w_1 \end{pmatrix}$$

Clearly if $k + 5 \neq 0$, i.e., $k \neq -5$, there will be a unique solution for x_1, x_2, and x_3 for any vector (w_1, w_2, w_3). Thus if $k \neq 5$, the range of the linear operator will be $\mathbb{R}^3$. But if $k = -5$, the third row of the augmented matrix leads to the equation

$$0x_1 + 0x_2 + 0x_3 = w_3 + 2w_2 - 3w_1$$

which has the solution only if $w_3 + 2w_2 - 3w_1 = 0$. So if $k = 5$ and $w_3 + 2w_2 - 3w_1 \neq 0$, the linear operator defined by the given equations does not have any image in $\mathbb{R}^3$.
When the map is the line $w_1 = 1 + 2t, w_2 = 1 + t, w_3 = 1 + 4t$, the augmented matrix becomes

$$\begin{pmatrix} 1 & 2 & 1 & 1+2t \\ 0 & 5 & 6 & 3+5t \\ 0 & 0 & k+5 & 0 \end{pmatrix}$$

Now if $k \neq -5$, there is a unique solution for (x_1, x_2, x_3), namely

$$x_3 = 0,\ x_2 = \tfrac{1}{5}(3 + 5t),\ x_1 = (1 + 2t) - \tfrac{2}{5}(3 + 5t) = -\tfrac{1}{5}$$

Hence when $k \neq -5$, the line $(x_1, x_2, x_3) = (-\frac{1}{5}, \frac{3}{5}, 0) + t(0, 1, 0)$ maps into the line $(w_1, w_2, w_3) = (1, 1, 1) + t(2, 1, 4)$ irrespective of the value of k (other than -5). But if $k = -5$, we have the following equations

$$x_1 + 2x_2 + x_3 = 1 + 2t,\ 5x_2 + 6x_3 = 3 + 5t,\ 0 = 0$$

t can be readily eliminated between the above two equations by equating the value of t. We have

$$t = \tfrac{1}{2}(x_1 + 2x_2 + x_3 - 1) = \tfrac{1}{5}(5x_2 + 6x_3 - 3)$$
$$\Rightarrow 5(x_1 + 2x_2 + x_3 - 1) = 2(5x_2 + 6x_3 - 3)$$
$$\Rightarrow 5x_1 + 10x_2 + 5x_3 - 5 = 10x_2 + 12x_3 - 6$$
$$\Rightarrow 5x_1 - 7x_3 + 1 = 0$$

So when $k = -5$, it is the plane $5x_1 - 7x_3 + 1 = 0$ (and not just the line $(x_1, x_2, x_3) = (-\frac{1}{5}, \frac{3}{5}, 0) + t(0, 1, 0)$) that maps into the given line $(w_1, w_2, w_3) = (1, 1, 1) + t(2, 1, 4)$.

4. If V is a set of ordered pairs (x, y) of real numbers with the following operations.

$(x,\ y) + (x',\ y') = (x + x',\ y + y' + 1)$ and $k(x,\ y) = (kx,\ ky + k - 1)$,
determine if it is a vector space. If it is not, list all axioms that fail to hold.

Solution:

Let, in the following, $\mathbf{u} = (x,\ y),\ \mathbf{v} = (x',\ y'),\ \mathbf{w} = (x'',\ y'')$, and k and l be the scalar numbers.

Axiom 1: Clearly $\mathbf{u}+\mathbf{v} = (x+x',\ y+y'+1)$ is an ordered pair of real number, and therefore belongs to V. So Axiom 1 holds.

Axiom 2: $\mathbf{u} + \mathbf{v} = (x + x',\ y + y' + 1),\ \mathbf{v} + \mathbf{u} = (x' + x,\ y' + y + 1) = \mathbf{u} + \mathbf{v}$, so Axiom 2 holds.

Axiom 3: $\mathbf{u}+\mathbf{v} = (x+x',\ y+y'+1),\ (\mathbf{u}+\mathbf{v})+\mathbf{w} = (x+x',\ y+y'+1)+(x'',\ y'') = (x + x' + x'',\ y + y' + 1 + y'' + 1) = (x + x' + x'',\ y + y' + y'' + 2)$

$\mathbf{v}+\mathbf{w} = (x'+x'',\ y'+y''+1),\ \mathbf{u}+(\mathbf{v}+\mathbf{w}) = (x,\ y)+(x'+x'',\ y'+y''+1) = (x + x' + x'',\ y + y' + y'' + 1 + 1) = (x + x' + x'',\ y + y' + y'' + 2)$,

so Axiom 3 holds.

Axiom 4: Let $\mathbf{e} = (z, w)$ be an additive identity, then $\mathbf{u} + \mathbf{e} = \mathbf{e} + \mathbf{u} = \mathbf{u}$

$\Rightarrow (x,\ y) + (z,\ w) = (x,\ y) \Rightarrow (x + z,\ y + w + 1) = (x,\ y)$

$\Rightarrow x + z = x,\ y + w + 1 = y \Rightarrow z = 0,\ w = -1$

Hence (0, -1) is an additive identity of V.

Axiom 5: Let $\mathbf{v}$ be an additive inverse of $\mathbf{u}$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} = \mathbf{e}$

$\Rightarrow (x,\ y) + (x',\ y') = (0, -1) \Rightarrow (x + x',\ y + y' + 1) = (0, -1)$

$\Rightarrow x + x' = 0,\ y + y' + 1 = -1 \Rightarrow x' = -x,\ y' = -y - 2$

Hence $(-x, -y - 2)$ is an additive inverse of (x, y), and Axiom 5 holds.

Axiom 6: Obviously $k\mathbf{u} = (kx,\ ky + k - 1)$ is an ordered pair of real number, and therefore belongs to V. So Axiom 6 holds.

Axiom 7: $k(\mathbf{u}+\mathbf{v}) = k(x+x',\ y+y'+1) = (k(x+x'),\ k(y+y'+1)+k-1) = (k(x + x'),\ k(y + y' + 2) - 1)$

$k\mathbf{u} + k\mathbf{v} = k(x,\ y) + k(x',\ y') = (kx,\ ky + k - 1) + (kx',\ ky' + k - 1) = (kx + ky,\ ky + k - 1 + ky' + k - 1 + 1) = (k(x + x'),\ k(y + y' + 2) - 1)$

so Axiom 7 holds.

Axiom 8: $(k + l)\mathbf{u} = (k + l)(x, y) = ((k + l)x,\ (k + l)y + k + l - 1)$

$k\mathbf{u}+l\mathbf{u} = k(x, y)+l(x, y) = (kx, ky+k-1)+(lx, ly+l-1) = (kx+lx,\ ky+k - 1 + ly + l - 1 + 1) = ((k + l)x,\ (k + l)y + k + l - 1)$

so Axiom 8 holds.

Axiom 9: $k(l\mathbf{u}) = k(l(x, y)) = k(lx, ly + l - 1) = (klx,\ k(ly + l - 1) + k - 1) = (klx, kl(y + 1) - 1)$

$kl\mathbf{u} = kl(x, y) = (klx,\ kly + kl - 1) = (klx, kl(y + 1) - 1)$, so Axiom 9 holds.

Axiom 10. $1\mathbf{u} = 1(x, y) = (x, y + 1 - 1) = (x, y) = \mathbf{u}$, so Axiom 10 holds.

Since all the ten axioms are satisfied, therefore V is a vector space.

5. Is the set V of all 2 x 2 matrices with equal column sums a subspace of M_{22}? If not, why not?

Solution:

Let $\mathbf{u}$, $\mathbf{v} \in V$, and k be any scalar. We take

$$\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

so that, because of the equal column sum property of V,

$$a + c = b + d, \text{ and } p + r = q + s. \qquad (*)$$

Now $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ clearly belongs to V $(0 + 0 = 0 + 0)$, so V is not empty.

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} a + p & b + q \\ c + r & d + s \end{pmatrix}$$

But $a + p + c + r = b + q + d + s$ (On adding the two equations in (*)), hence $\mathbf{u} + \mathbf{v}$ belongs to V, and the closure property is satisfied for addition.

$$k\mathbf{u} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

Obviously $ka+kc = kb+kd$ (On muliplying the first equation in (*) by k), hence $k\mathbf{u}$ belongs to V, so the closure property is satisfied for scalar multiplication. Since all the three requirements are satisfied for subspace, V is a subspace of M_{22}.