Quiz 2: ch4, 6-8, 11
ch16: Greedy Algorithms

Open book, paper notes
No electronic devices
Please show your work

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CMPT231
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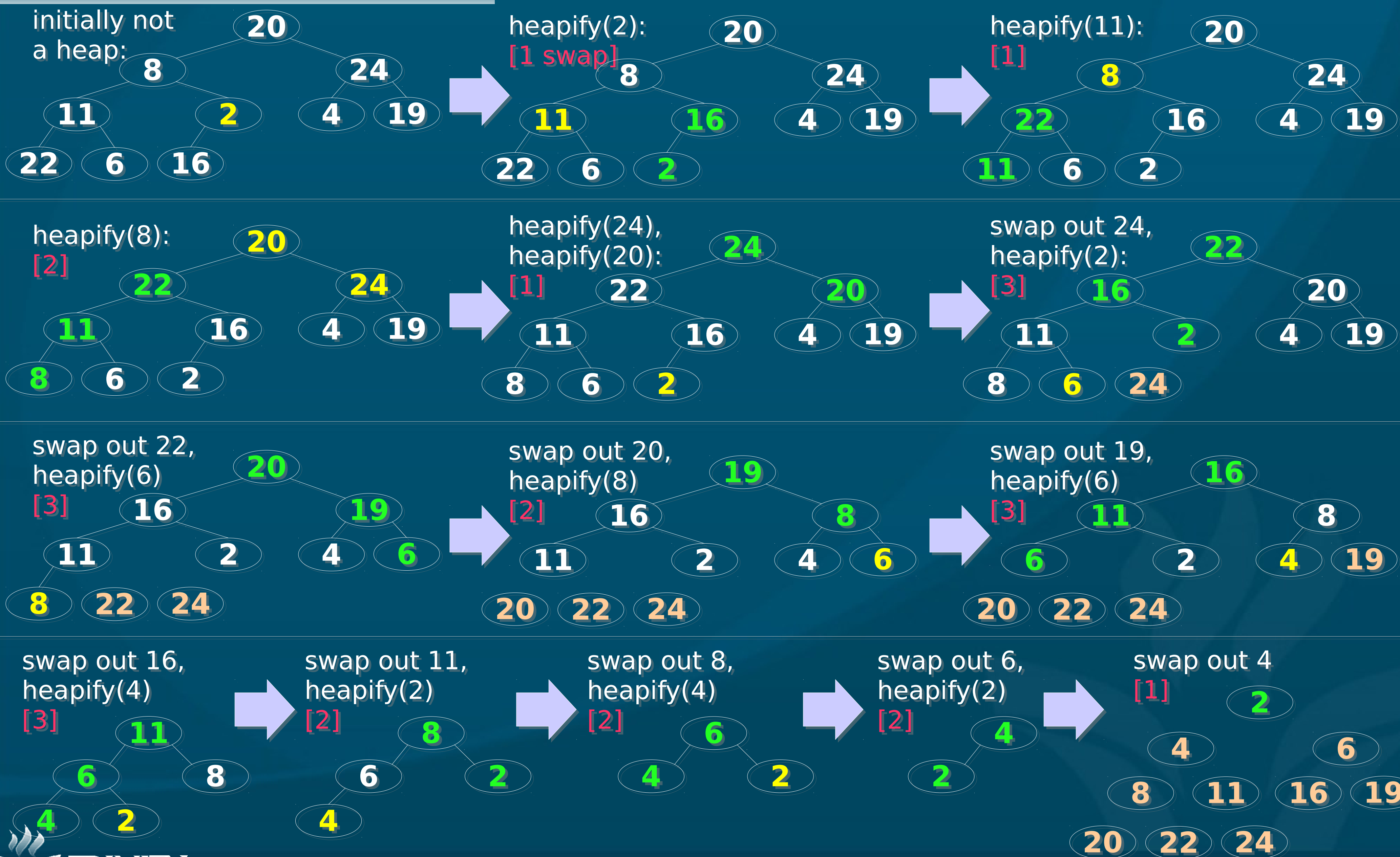
Quiz 2: 30pts

- Input for all: [20, 8, 24, 11, 2, 4, 19, 22, 6, 16]

- **[10]** Demonstrate each step of Heapsort on the input. How many non-trivial swaps are performed?
- **[10]** Demonstrate each step of (non-randomised) Quicksort on the input. How many non-trivial swaps?
- **[10]** Demonstrate radix sort on the input with base-4 digits.

Quiz 2: solutions #1



Quiz 2: solutions #1-2

- Heapsort: 26 swaps
- Quicksort: only 12 swaps!

Pivot=16:	20	8	24	11	2	4	19	22	6	16	Swaps
.	8	20	.	.	.	.	.	.	.	.	1
.	8	11	24	20	.	.	.	.	.	.	1
.	8	11	2	20	24	.	.	.	.	.	1
.	8	11	2	4	24	20	.	.	.	.	1
.	8	11	2	4	6	20	19	22	24	.	1
.	8	11	2	4	6	16	19	22	24	20	1
6:	8	11	2	4	6	16	19	22	24	20	.
2:	2	4	6	11	8	.	.	.	.	.	3
4:	2	4	.	.	.	.	.	.	.	.	.
8:	.	.	.	8	11	.	.	.	.	.	1
20:	.	.	.	.	.	.	19	20	24	22	1
22:	.	.	.	.	.	.	.	.	22	24	1

Quiz 2: solutions #3

20
8
24
11
2
4
19
22
6
16

base 4 →

1	1	0
0	2	0
1	2	0
0	2	3
0	0	2
0	1	0
1	0	3
1	1	2
0	1	2
1	0	0

sort 1 →

1	1	0
0	2	0
1	2	0
0	1	0
1	0	0
0	0	2
1	1	2
0	1	2
0	2	3
1	0	3

sort 4 →

1	0	0
0	0	2
1	0	3
1	1	0
0	1	0
1	1	2
0	1	2
0	2	0
1	2	0
0	2	3

sort 16 →

0	0	2
0	1	0
0	1	2
0	2	0
0	2	3
1	0	0
1	0	3
1	1	0
1	1	2
1	2	0

decimal →

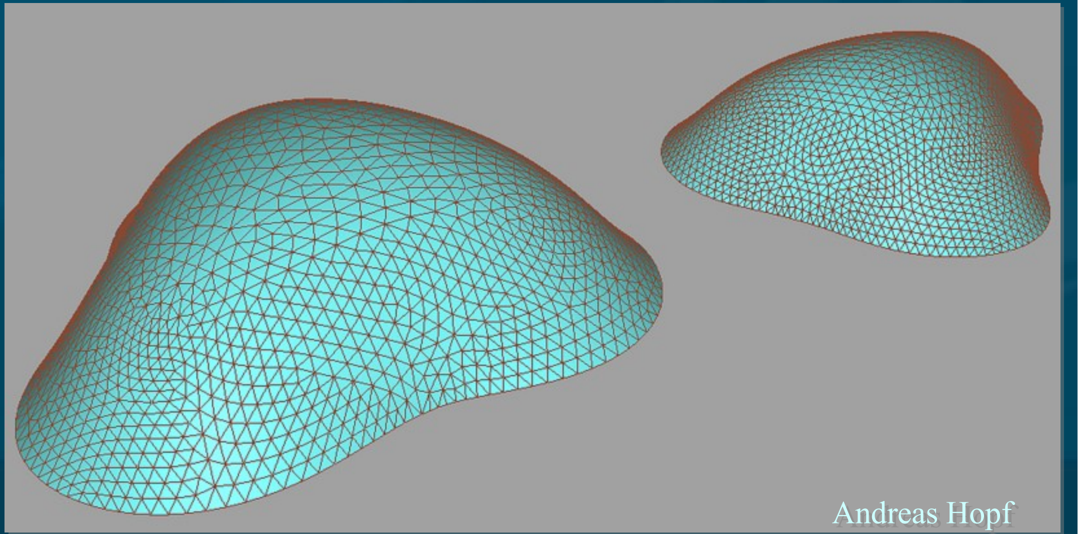
2
4
6
8
11
16
19
20
22
24

Outline for today

- Greedy algorithms
  - Example: Activity selection
  - Optimal substructure
    - Naive recursive solution
  - Greedy choice property
    - Recursive greedy solution
    - Iterative greedy solution
  - Example: Knapsack problems
  - Example: Huffman coding

Greedy algorithms

- Another approach to optimisation
  - Faster than dynamic programming, when applicable
- At each decision point, go for immediate gains
  - Locally optimal choices ⇒ global optimum
- Not all problems have optimal substructure
  - Hybrid optimisation strategies use large jumps to get to right “hill”, then greedy “hill-climbing” to get to the top



Andreas Hopf

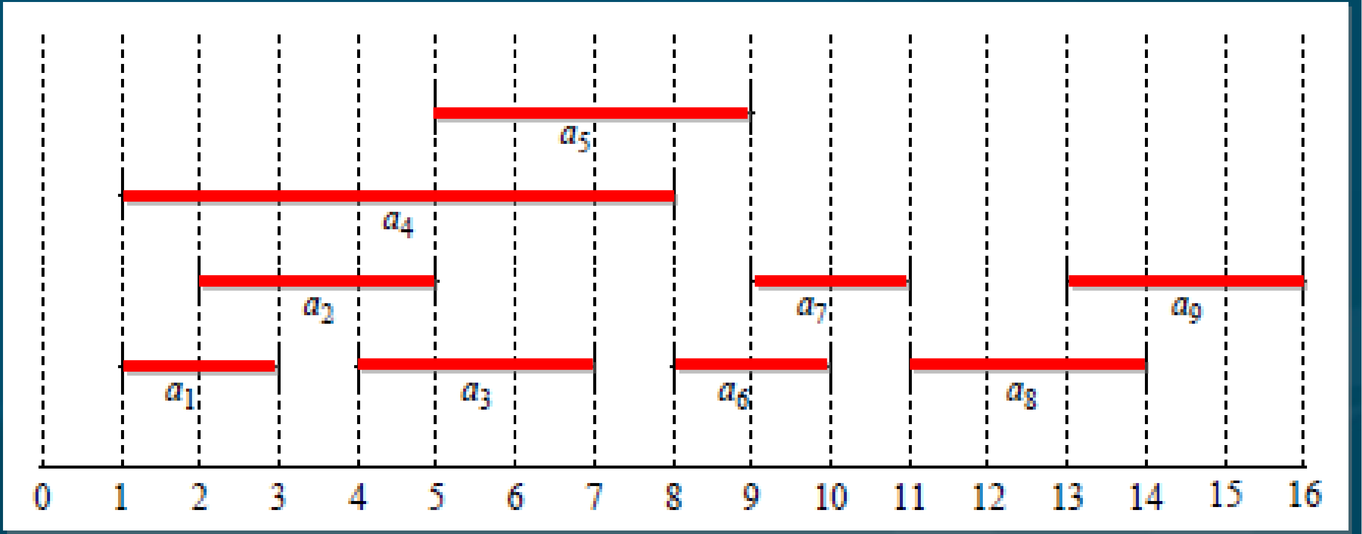
Problem-solving outline

- Find optimal substructure (e.g., recurrence)
- Convert to naïve recursive solution (code)
  - Could then be converted to dynamic prog.
- Use greedy choice to simplify the recurrence so only one subproblem remains
  - Don't have to iterate through all subproblems
  - Prove greedy choice yields global optimum!
- Convert to recursive greedy solution
- Convert to iterative greedy solution

Example: activity selection

i	s	f
1	1	3
2	2	5
3	4	7
4	1	8
5	5	9
6	8	10
7	9	11
8	11	14
9	13	16

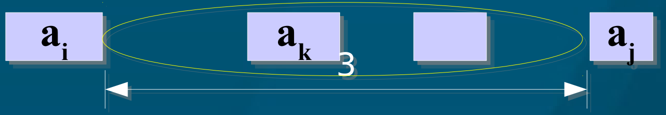
- Activities S = {a_1, ..., a_n} which each require exclusive use of a shared resource
  - Each activity has start/finish times [s_i, f_i)
  - Activities are sorted by finish times
- ⇒ Find largest subset of S where all activities are non-overlapping
- e.g., a_2 and a_5 do not overlap:



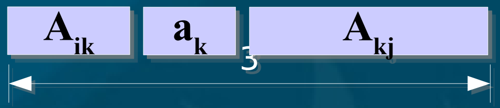
Solutions?

Solving: optimal substructure

- Let $S_{ij} = \{a_k \in S: f_i \le s_k < f_k \le s_j\}$: all activities that start after f_i and finish before s_j



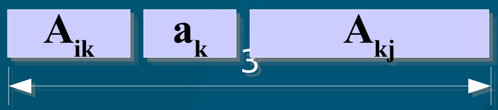
  - Any activity in S_{ij} will be compatible with:
    - Any activity that finishes by f_i
    - Any activity that starts no earlier than s_j
- Let A_{ij} be a solution for S_{ij}:
a largest mutually-compatible subset of activities



- Pick an activity $a_k \in A_{ij}$, and partition A_{ij} into
  - $A_{ik} = A_{ij} \cap S_{ik}$: those that finish before a_k starts
  - $A_{kj} = A_{ij} \cap S_{kj}$: those that start after a_k finishes

Proof of optimal substructure

- Claim: A_{ik} and A_{kj} are optimal solutions for S_{ik}, S_{kj}
- Proof (for A_{ik}): assume not:



  - Let A'_{ik} be a better solution:
    non-overlapping elements, and $|A'_{ik}| > |A_{ik}|$.
  - Then $A'_{ik} \cup \{a_k\} \cup A_{kj}$ would be a solution for S_{ij}, and its size is larger than $A_{ij} = A_{ik} \cup \{a_k\} \cup A_{kj}$.
  - Contradicts the premise that A_{ij} was optimal.
- ⇒ Optimal substructure: split on a_k,
  recurse twice on S_{ik} and S_{kj},
  iterate over all choices of a_k and pick the best

Naive recursive solution

- Let $c[i,j]$ = size of optimal solution for S_{ij}:
  - Splitting on a_k yields $c[i,j] = c[i,k] + 1 + c[k,j]$
  - Which choice of a_k is best? Naive: try all
- Recurrence: $c[i,j] = \max_{a\_k \in S\_ij} (c[i,k] + 1 + c[k,j])$
  - Base case: if $S_{ij} = \emptyset$, then $c[i,j] = 0$
- Could implement this using dynamic programming
  - Fill in 2D table for $c[i,j]$, bottom-up
  - Auxiliary table storing the solutions A_{ij}
- With this problem, though, we can do better!

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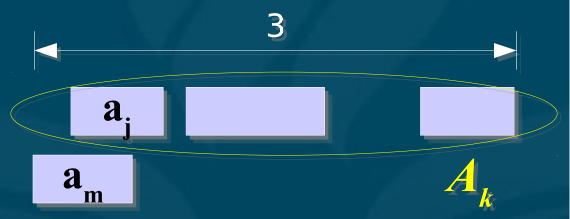
  - Example: Knapsack problems
  - Example: Huffman coding

Greedy choice

- Which choice of a_k leaves as much as possible of the resource available for other activities?
  - One which finishes the earliest
  - Since activities are sorted by finish time, just choose the first activity!
- Recurrence simplifies: to find optimal subset of S_{kj}, include a_k, then recurse on $S_k = \{a_i: s_i \geq f_k\}$: those that start after a_k finishes
  - Don't need to iterate over all choices of a_k
- We need to prove the greedy choice is optimal

Proof of greedy choice

- Let $S_k \neq \emptyset$ with $a_m \in S_k$ having earliest finish time.
  - Claim: $\exists$ optimal soln for S_k which includes a_m.
- Proof: Let A_k be an optimal solution for S_k.
  - If it includes a_m, then we're done.
- If not, let a_j be the first in A_k to finish.
  - Swap out a_m for a_j: let $A'_k = A_k - \{a_j\} \cup \{a_m\}$.
- Then A'_k is an optimal solution for S_k:
  - Size is same as A_k, and
  - Elements are non-overlapping: $f_m \leq f_j$



Recursive greedy solution

- Input: arrays s[], f[], with f[] sorted
  - Add a dummy entry f[0] = 0, so that $S_0 = S$.
- For each recursive subproblem S_k,
  - Skip over activities that overlap with a_k
  - Include the first activity that doesn't overlap, and recurse on the rest:
    - **def ActivitySel(s, f, k, n):**
      - **for m in k+1 .. n:**
        - **if (s[m] ≥ f[k]):**
          - **return {a_m} ∪ ActivitySel(s, f, m, n)**
      - **return NULL**
  - Initial call: ActivitySel(s, f, 0, n). (Θ(n)!)

Iterative greedy solution

- Recursive solution is nearly tail-recursive, easy to convert to more efficient iterative solution:
    - **def ActivitySel(s, f):**
        - $A = \{a_1\}$
        - $k = 1$
        - **for** m **in** $2 .. \text{length}(f)$:
            - **if** $(s[m] \geq f[k])$:
                - $A = A \cup \{a_m\}$
                - $k = m$
        - **return** A
- Complexity: $\Theta(n)$
    - If need to pre-sort on f[], then $\Theta(n \lg n)$

i	s	f
1	1	3
2	2	5
3	4	7
4	1	8
5	5	9
6	8	10
7	9	11
8	11	14
9	13	16

Greedy vs dynamic prog.

- Dynamic prog. more general
  - Not all problems have greedy property
- Dynamic prog. fills in table bottom-up
  - Greedy choice done top-down
- Choice in dyn. prog. needs all smaller subprobs
  - Greedy choice is simpler, so can make choice before solving subproblem
- Proving the greedy property:
  - Assume an optimal solution
  - Modify it to include the greedy choice
  - Show that it's still optimal

Optimising for greedy choice

- Often need to pre-process input to make the greedy choice easier
  - Sorted activities by finish time
  - Greedy choice can be done in O(1) each time
  - Sorting takes O(n lg n)
- If input is dynamically generated (can't sort whole list in advance), then
  - Priority queue: pop the most optimal choice

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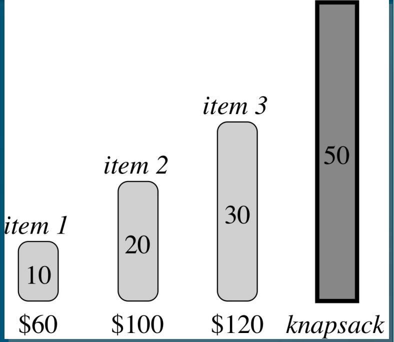
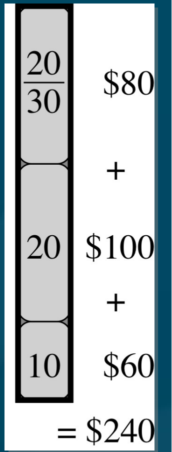
Knapsack problem

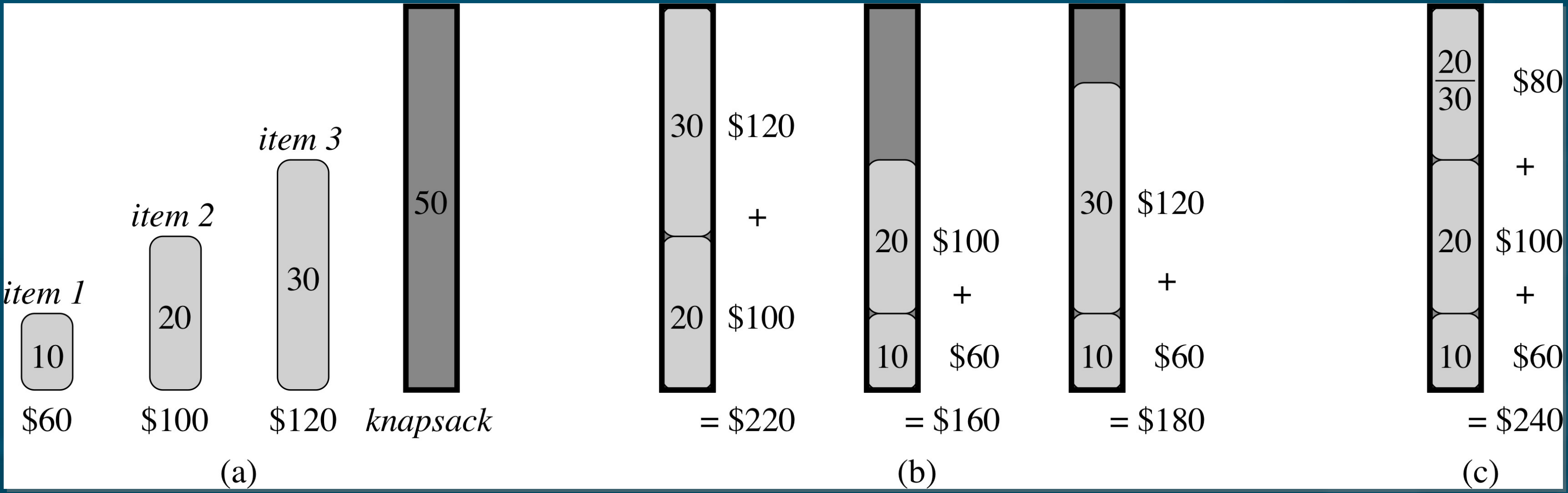


- Fractional knapsack problem:
  - n items, each with weight w_i and value v_i.
  - Maximise total value, subject to total weight W
  - Can take fractions of an item (think of liquids)
- Greedy soln: sort items by value-to-weight ratio
  - Greedy choice: take item with largest v_i / w_i.
  - Last spot may be filled with fractional item
    - def FractionalKnapsack(v, w, W):
      - while totwt < W:
        - add next item in decreasing order of value-to-weight
      - replace last item with 1-(totwt-W) of itself



0-1 Knapsack

- Variant that does not allow fractions of an item
- Greedy strategy no longer works!
- Making initial locally-optimal choices locks us out of making later globally-optimal choices
- Still possible to solve using dynamic programming (Ex 16.2-2)



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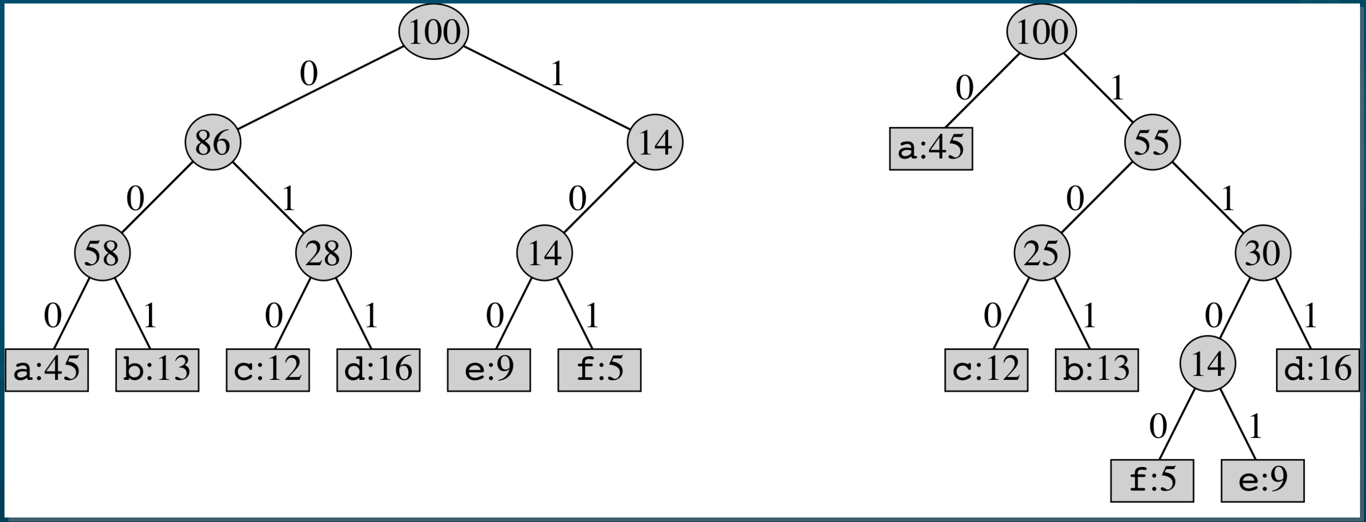
Encoding

	a	b	c	d	e	f
Frequency (in thousands)	45	13	12	16	9	5
Fixed-length codeword	000	001	010	011	100	101
Variable-length codeword	0	101	100	111	1101	1100

- Given a text with a known set of characters
  - Encode each character with a binary codeword
- Fixed-length code: all codewords same length
  - “cafe” ⇒ 010 000 101 100
- Variable-length code: some codes lower cost
  - “cafe” ⇒ 100 0 1100 1101
  - Compression: choose shorter codes for more frequent characters
- Prefix code: no code is a prefix of another
  - Unique parsing; don't need to delimit chars
  - “cafe” ⇒ 100011001101

Code trees

- Prefix code ⇒ code tree: binary tree where nodes represent prefixes; characters are at leaves
  - Fixed-length code ⇒ leaves all at same level
  - Decoding = walk down the tree
    - Cost of a char = depth in tree
- Total cost of encoding a file using a given tree:
  - Σ_c [freq(c) * depth(c)]



Huffman coding

- Build tree bottom-up
  - Start with two least-common chars
  - Merge to make new subtree with combined freq
- Min-priority queue manages the greedy choice
- Input: array of char nodes with .freq attribs

```
def huffman(chars):
  • Q = new MinQueue(chars)
  • for i in 1 .. length(chars)-1:
      • z = new Node
      • z.left = Q.popmin()
      • z.right = Q.popmin()
      • z.freq = z.left.freq + z.right.freq
      • Q.push(z)
  • return Q.popmin()
```

char	freq
a	15
b	5
c	9
d	7
e	18
f	10